PURE AND APPLIED MATHEMATICS

A DEKKER SERIES OF MONOGRAPHS AND TEXTBOOKS

STOCHASTIC VERSUS DETERMINISTIC SYSTEMS OF DIFFERENTIAL EQUATIONS

G. S. LADDE

M. SAMBANDHAM

STOCHASTIC VERSUS DETERMINISTIC SYSTEMS OF DIFFERENTIAL EQUATIONS

G. S. LADDE
The University of Texas at Arlington
Arlington, Texas, U.S.A.

M. SAMBANDHAM
Morehouse College
Atlanta, Georgia, U.S.A.

MARCEL DEKKER

MARCEL DEKKER, INC. NEW YORK • BASEL

Library of Congress Cataloging-in-Publication Data
A catalog record for this book is available from the Library of Congress.

ISBN: 0-8247-4697-X

This book is printed on acid-free paper.

Headquarters
Marcel Dekker, Inc.
270 Madison Avenue, New York, NY 10016, U.S.A.
tel: 212-696-9000; fax: 212-685-4540

Distribution and Customer Service
Marcel Dekker, Inc.
Cimarron Road, Monticello, New York 12701, U.S.A.
tel: 800-228-1160; fax: 845-796-1772

Eastern Hemisphere Distribution
Marcel Dekker AG
Hutgasse 4, Postfach 812, CH-4001 Basel, Switzerland
tel: 41-61-260-6300; fax: 41-61-260-6333

World Wide Web
http://www.dekker.com

The publisher offers discounts on this book when ordered in bulk quantities. For more information, write to Special Sales/Professional Marketing at the headquarters address above.

Current printing (last digit):

10 9 8 7 6 5 4 3 2 1

PRINTED IN THE UNITED STATES OF AMERICA

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

MONOGRAPHS AND TEXTBOOKS IN PURE AND APPLIED MATHEMATICS

1. *K. Yano,* Integral Formulas in Riemannian Geometry (1970)
2. *S. Kobayashi,* Hyperbolic Manifolds and Holomorphic Mappings (1970)
3. *V. S. Vladimirov,* Equations of Mathematical Physics (A. Jeffrey, ed.; A. Littlewood, trans.) (1970)
4. *B. N. Pshenichnyi,* Necessary Conditions for an Extremum (L. Neustadt, translation ed.; K. Makowski, trans.) (1971)
5. *L. Narici et al.,* Functional Analysis and Valuation Theory (1971)
6. *S. S. Passman,* Infinite Group Rings (1971)
7. *L. Dornhoff,* Group Representation Theory. Part A: Ordinary Representation Theory. Part B: Modular Representation Theory (1971, 1972)
8. *W. Boothby and G. L. Weiss, eds.,* Symmetric Spaces (1972)
9. *Y. Matsushima,* Differentiable Manifolds (E. T. Kobayashi, trans.) (1972)
10. *L. E. Ward, Jr.,* Topology (1972)
11. *A. Babakhanian,* Cohomological Methods in Group Theory (1972)
12. *R. Gilmer,* Multiplicative Ideal Theory (1972)
13. *J. Yeh,* Stochastic Processes and the Wiener Integral (1973)
14. *J. Barros-Neto,* Introduction to the Theory of Distributions (1973)
15. *R. Larsen,* Functional Analysis (1973)
16. *K. Yano and S. Ishihara,* Tangent and Cotangent Bundles (1973)
17. *C. Procesi,* Rings with Polynomial Identities (1973)
18. *R. Hermann,* Geometry, Physics, and Systems (1973)
19. *N. R. Wallach,* Harmonic Analysis on Homogeneous Spaces (1973)
20. *J. Dieudonné,* Introduction to the Theory of Formal Groups (1973)
21. *I. Vaisman,* Cohomology and Differential Forms (1973)
22. *B.-Y. Chen,* Geometry of Submanifolds (1973)
23. *M. Marcus,* Finite Dimensional Multilinear Algebra (in two parts) (1973, 1975)
24. *R. Larsen,* Banach Algebras (1973)
25. *R. O. Kujala and A. L. Vitter, eds.,* Value Distribution Theory: Part A; Part B: Deficit and Bezout Estimates by Wilhelm Stoll (1973)
26. *K. B. Stolarsky,* Algebraic Numbers and Diophantine Approximation (1974)
27. *A. R. Magid,* The Separable Galois Theory of Commutative Rings (1974)
28. *B. R. McDonald,* Finite Rings with Identity (1974)
29. *J. Satake,* Linear Algebra (S. Koh et al., trans.) (1975)
30. *J. S. Golan,* Localization of Noncommutative Rings (1975)
31. *G. Klambauer,* Mathematical Analysis (1975)
32. *M. K. Agoston,* Algebraic Topology (1976)
33. *K. R. Goodearl,* Ring Theory (1976)
34. *L. E. Mansfield,* Linear Algebra with Geometric Applications (1976)
35. *N. J. Pullman,* Matrix Theory and Its Applications (1976)
36. *B. R. McDonald,* Geometric Algebra Over Local Rings (1976)
37. *C. W. Groetsch,* Generalized Inverses of Linear Operators (1977)
38. *J. E. Kuczkowski and J. L. Gersting,* Abstract Algebra (1977)
39. *C. O. Christenson and W. L. Voxman,* Aspects of Topology (1977)
40. *M. Nagata,* Field Theory (1977)
41. *R. L. Long,* Algebraic Number Theory (1977)
42. *W. F. Pfeffer,* Integrals and Measures (1977)
43. *R. L. Wheeden and A. Zygmund,* Measure and Integral (1977)
44. *J. H. Curtiss,* Introduction to Functions of a Complex Variable (1978)
45. *K. Hrbacek and T. Jech,* Introduction to Set Theory (1978)
46. *W. S. Massey,* Homology and Cohomology Theory (1978)
47. *M. Marcus,* Introduction to Modern Algebra (1978)
48. *E. C. Young,* Vector and Tensor Analysis (1978)
49. *S. B. Nadler, Jr.,* Hyperspaces of Sets (1978)
50. *S. K. Segal,* Topics in Group Kings (1978)
51. *A. C. M. van Rooij,* Non-Archimedean Functional Analysis (1978)
52. *L. Corwin and R. Szczarba,* Calculus in Vector Spaces (1979)
53. *C. Sadosky,* Interpolation of Operators and Singular Integrals (1979)
54. *J. Cronin,* Differential Equations (1980)
55. *C. W. Groetsch,* Elements of Applicable Functional Analysis (1980)
56. *I. Vaisman,* Foundations of Three-Dimensional Euclidean Geometry (1980)
57. *H. I. Freedan,* Deterministic Mathematical Models in Population Ecology (1980)

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

58. *S. B. Chae,* Lebesgue Integration (1980)
59. *C. S. Rees et al.,* Theory and Applications of Fourier Analysis (1981)
60. *L. Nachbin,* Introduction to Functional Analysis (R. M. Aron, trans.) (1981)
61. *G. Orzech and M. Orzech,* Plane Algebraic Curves (1981)
62. *R. Johnsonbaugh and W. E. Pfaffenberger,* Foundations of Mathematical Analysis (1981)
63. *W. L. Voxman and R. H. Goetschel,* Advanced Calculus (1981)
64. *L. J. Corwin and R. H. Szczarba,* Multivariable Calculus (1982)
65. *V. I. Istrăţescu,* Introduction to Linear Operator Theory (1981)
66. *R. D. Järvinen,* Finite and Infinite Dimensional Linear Spaces (1981)
67. *J. K. Beem and P. E. Ehrlich,* Global Lorentzian Geometry (1981)
68. *D. L. Armacost,* The Structure of Locally Compact Abelian Groups (1981)
69. *J. W. Brewer and M. K. Smith, eds.,* Emmy Noether: A Tribute (1981)
70. *K. H. Kim,* Boolean Matrix Theory and Applications (1982)
71. *T. W. Wieting,* The Mathematical Theory of Chromatic Plane Ornaments (1982)
72. *D. B. Gauld,* Differential Topology (1982)
73. *R. L. Faber,* Foundations of Euclidean and Non-Euclidean Geometry (1983)
74. *M. Carmeli,* Statistical Theory and Random Matrices (1983)
75. *J. H. Carruth et al.,* The Theory of Topological Semigroups (1983)
76. *R. L. Faber,* Differential Geometry and Relativity Theory (1983)
77. *S. Barnett,* Polynomials and Linear Control Systems (1983)
78. *G. Karpilovsky,* Commutative Group Algebras (1983)
79. *F. Van Oystaeyen and A. Verschoren,* Relative Invariants of Rings (1983)
80. *I. Vaisman,* A First Course in Differential Geometry (1984)
81. *G. W. Swan,* Applications of Optimal Control Theory in Biomedicine (1984)
82. *T. Petrie and J. D. Randall,* Transformation Groups on Manifolds (1984)
83. *K. Goebel and S. Reich,* Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings (1984)
84. *T. Albu and C. Năstăsescu,* Relative Finiteness in Module Theory (1984)
85. *K. Hrbacek and T. Jech,* Introduction to Set Theory: Second Edition (1984)
86. *F. Van Oystaeyen and A. Verschoren,* Relative Invariants of Rings (1984)
87. *B. R. McDonald,* Linear Algebra Over Commutative Rings (1984)
88. *M. Namba,* Geometry of Projective Algebraic Curves (1984)
89. *G. F. Webb,* Theory of Nonlinear Age-Dependent Population Dynamics (1985)
90. *M. R. Bremner et al.,* Tables of Dominant Weight Multiplicities for Representations of Simple Lie Algebras (1985)
91. *A. E. Fekete,* Real Linear Algebra (1985)
92. *S. B. Chae,* Holomorphy and Calculus in Normed Spaces (1985)
93. *A. J. Jerri,* Introduction to Integral Equations with Applications (1985)
94. *G. Karpilovsky,* Projective Representations of Finite Groups (1985)
95. *L. Narici and E. Beckenstein,* Topological Vector Spaces (1985)
96. *J. Weeks,* The Shape of Space (1985)
97. *P. R. Gribik and K. O. Kortanek,* Extremal Methods of Operations Research (1985)
98. *J.-A. Chao and W. A. Woyczynski, eds.,* Probability Theory and Harmonic Analysis (1986)
99. *G. D. Crown et al.,* Abstract Algebra (1986)
100. *J. H. Carruth et al.,* The Theory of Topological Semigroups, Volume 2 (1986)
101. *R. S. Doran and V. A. Belfi,* Characterizations of C*-Algebras (1986)
102. *M. W. Jeter,* Mathematical Programming (1986)
103. *M. Altman,* A Unified Theory of Nonlinear Operator and Evolution Equations with Applications (1986)
104. *A. Verschoren,* Relative Invariants of Sheaves (1987)
105. *R. A. Usmani,* Applied Linear Algebra (1987)
106. *P. Blass and J. Lang,* Zariski Surfaces and Differential Equations in Characteristic $p > 0$ (1987)
107. *J. A. Reneke et al.,* Structured Hereditary Systems (1987)
108. *H. Busemann and B. B. Phadke,* Spaces with Distinguished Geodesics (1987)
109. *R. Harte,* Invertibility and Singularity for Bounded Linear Operators (1988)
110. *G. S. Ladde et al.,* Oscillation Theory of Differential Equations with Deviating Arguments (1987)
111. *L. Dudkin et al.,* Iterative Aggregation Theory (1987)
112. *T. Okubo,* Differential Geometry (1987)
113. *D. L. Stancl and M. L. Stancl,* Real Analysis with Point-Set Topology (1987)
114. *T. C. Gard,* Introduction to Stochastic Differential Equations (1988)
115. *S. S. Abhyankar,* Enumerative Combinatorics of Young Tableaux (1988)
116. *H. Strade and R. Farnsteiner,* Modular Lie Algebras and Their Representations (1988)

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

117. *J. A. Huckaba,* Commutative Rings with Zero Divisors (1988)
118. *W. D. Wallis,* Combinatorial Designs (1988)
119. *W. Więsław,* Topological Fields (1988)
120. *G. Karpilovsky,* Field Theory (1988)
121. *S. Caenepeel and F. Van Oystaeyen,* Brauer Groups and the Cohomology of Graded Rings (1989)
122. *W. Kozlowski,* Modular Function Spaces (1988)
123. *E. Lowen-Colebunders,* Function Classes of Cauchy Continuous Maps (1989)
124. *M. Pavel,* Fundamentals of Pattern Recognition (1989)
125. *V. Lakshmikantham et al.,* Stability Analysis of Nonlinear Systems (1989)
126. *R. Sivaramakrishnan,* The Classical Theory of Arithmetic Functions (1989)
127. *N. A. Watson,* Parabolic Equations on an Infinite Strip (1989)
128. *K. J. Hastings,* Introduction to the Mathematics of Operations Research (1989)
129. *B. Fine,* Algebraic Theory of the Bianchi Groups (1989)
130. *D. N. Dikranjan et al.,* Topological Groups (1989)
131. *J. C. Morgan II,* Point Set Theory (1990)
132. *P. Biler and A. Witkowski,* Problems in Mathematical Analysis (1990)
133. *H. J. Sussmann,* Nonlinear Controllability and Optimal Control (1990)
134. *J.-P. Florens et al.,* Elements of Bayesian Statistics (1990)
135. *N. Shell,* Topological Fields and Near Valuations (1990)
136. *B. F. Doolin and C. F. Martin,* Introduction to Differential Geometry for Engineers (1990)
137. *S. S. Holland, Jr.,* Applied Analysis by the Hilbert Space Method (1990)
138. *J. Okniński,* Semigroup Algebras (1990)
139. *K. Zhu,* Operator Theory in Function Spaces (1990)
140. *G. B. Price,* An Introduction to Multicomplex Spaces and Functions (1991)
141. *R. B. Darst,* Introduction to Linear Programming (1991)
142. *P. L. Sachdev,* Nonlinear Ordinary Differential Equations and Their Applications (1991)
143. *T. Husain,* Orthogonal Schauder Bases (1991)
144. *J. Foran,* Fundamentals of Real Analysis (1991)
145. *W. C. Brown,* Matrices and Vector Spaces (1991)
146. *M. M. Rao and Z. D. Ren,* Theory of Orlicz Spaces (1991)
147. *J. S. Golan and T. Head,* Modules and the Structures of Rings (1991)
148. *C. Small,* Arithmetic of Finite Fields (1991)
149. *K. Yang,* Complex Algebraic Geometry (1991)
150. *D. G. Hoffman et al.,* Coding Theory (1991)
151. *M. O. González,* Classical Complex Analysis (1992)
152. *M. O. González,* Complex Analysis (1992)
153. *L. W. Baggett,* Functional Analysis (1992)
154. *M. Sniedovich,* Dynamic Programming (1992)
155. *R. P. Agarwal,* Difference Equations and Inequalities (1992)
156. *C. Brezinski,* Biorthogonality and Its Applications to Numerical Analysis (1992)
157. *C. Swartz,* An Introduction to Functional Analysis (1992)
158. *S. B. Nadler, Jr.,* Continuum Theory (1992)
159. *M. A. Al-Gwaiz,* Theory of Distributions (1992)
160. *E. Perry,* Geometry: Axiomatic Developments with Problem Solving (1992)
161. *E. Castillo and M. R. Ruiz-Cobo,* Functional Equations and Modelling in Science and Engineering (1992)
162. *A. J. Jerri,* Integral and Discrete Transforms with Applications and Error Analysis (1992)
163. *A. Charlier et al.,* Tensors and the Clifford Algebra (1992)
164. *P. Biler and T. Nadzieja,* Problems and Examples in Differential Equations (1992)
165. *E. Hansen,* Global Optimization Using Interval Analysis (1992)
166. *S. Guerre-Delabrière,* Classical Sequences in Banach Spaces (1992)
167. *Y. C. Wong,* Introductory Theory of Topological Vector Spaces (1992)
168. *S. H. Kulkarni and B. V. Limaye,* Real Function Algebras (1992)
169. *W. C. Brown,* Matrices Over Commutative Rings (1993)
170. *J. Loustau and M. Dillon,* Linear Geometry with Computer Graphics (1993)
171. *W. V. Petryshyn,* Approximation-Solvability of Nonlinear Functional and Differential Equations (1993)
172. *E. C. Young,* Vector and Tensor Analysis: Second Edition (1993)
173. *T. A. Bick,* Elementary Boundary Value Problems (1993)
174. *M. Pavel,* Fundamentals of Pattern Recognition: Second Edition (1993)
175. *S. A. Albeverio et al.,* Noncommutative Distributions (1993)
176. *W. Fulks,* Complex Variables (1993)
177. *M. M. Rao,* Conditional Measures and Applications (1993)

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

178. *A. Janicki and A. Weron*, Simulation and Chaotic Behavior of α-Stable Stochastic Processes (1994)
179. *P. Neittaanmäki and D. Tiba*, Optimal Control of Nonlinear Parabolic Systems (1994)
180. *J. Cronin*, Differential Equations: Introduction and Qualitative Theory, Second Edition (1994)
181. *S. Heikkilä and V. Lakshmikantham*, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations (1994)
182. *X. Mao*, Exponential Stability of Stochastic Differential Equations (1994)
183. *B. S. Thomson*, Symmetric Properties of Real Functions (1994)
184. *J. E. Rubio*, Optimization and Nonstandard Analysis (1994)
185. *J. L. Bueso et al.*, Compatibility, Stability, and Sheaves (1995)
186. *A. N. Michel and K. Wang*, Qualitative Theory of Dynamical Systems (1995)
187. *M. R. Darnel*, Theory of Lattice-Ordered Groups (1995)
188. *Z. Naniewicz and P. D. Panagiotopoulos*, Mathematical Theory of Hemivariational Inequalities and Applications (1995)
189. *L. J. Corwin and R. H. Szczarba*, Calculus in Vector Spaces: Second Edition (1995)
190. *L. H. Erbe et al.*, Oscillation Theory for Functional Differential Equations (1995)
191. *S. Agaian et al.*, Binary Polynomial Transforms and Nonlinear Digital Filters (1995)
192. *M. I. Gil'*, Norm Estimations for Operation-Valued Functions and Applications (1995)
193. *P. A. Grillet*, Semigroups: An Introduction to the Structure Theory (1995)
194. *S. Kichenassamy*, Nonlinear Wave Equations (1996)
195. *V. F. Krotov*, Global Methods in Optimal Control Theory (1996)
196. *K. I. Beidar et al.*, Rings with Generalized Identities (1996)
197. *V. I. Arnautov et al.*, Introduction to the Theory of Topological Rings and Modules (1996)
198. *G. Sierksma*, Linear and Integer Programming (1996)
199. *R. Lasser*, Introduction to Fourier Series (1996)
200. *V. Sima*, Algorithms for Linear-Quadratic Optimization (1996)
201. *D. Redmond*, Number Theory (1996)
202. *J. K. Beem et al.*, Global Lorentzian Geometry: Second Edition (1996)
203. *M. Fontana et al.*, Prüfer Domains (1997)
204. *H. Tanabe*, Functional Analytic Methods for Partial Differential Equations (1997)
205. *C. Q. Zhang*, Integer Flows and Cycle Covers of Graphs (1997)
206. *E. Spiegel and C. J. O'Donnell*, Incidence Algebras (1997)
207. *B. Jakubczyk and W. Respondek*, Geometry of Feedback and Optimal Control (1998)
208. *T. W. Haynes et al.*, Fundamentals of Domination in Graphs (1998)
209. *T. W. Haynes et al., eds.*, Domination in Graphs: Advanced Topics (1998)
210. *L. A. D'Alotto et al.*, A Unified Signal Algebra Approach to Two-Dimensional Parallel Digital Signal Processing (1998)
211. *F. Halter-Koch*, Ideal Systems (1998)
212. *N. K. Govil et al., eds.*, Approximation Theory (1998)
213. *R. Cross*, Multivalued Linear Operators (1998)
214. *A. A. Martynyuk*, Stability by Liapunov's Matrix Function Method with Applications (1998)
215. *A. Favini and A. Yagi*, Degenerate Differential Equations in Banach Spaces (1999)
216. *A. Illanes and S. Nadler, Jr.*, Hyperspaces: Fundamentals and Recent Advances (1999)
217. *G. Kato and D. Struppa*, Fundamentals of Algebraic Microlocal Analysis (1999)
218. *G. X.-Z. Yuan*, KKM Theory and Applications in Nonlinear Analysis (1999)
219. *D. Motreanu and N. H. Pavel*, Tangency, Flow Invariance for Differential Equations, and Optimization Problems (1999)
220. *K. Hrbacek and T. Jech*, Introduction to Set Theory, Third Edition (1999)
221. *G. E. Kolosov*, Optimal Design of Control Systems (1999)
222. *N. L. Johnson*, Subplane Covered Nets (2000)
223. *B. Fine and G. Rosenberger*, Algebraic Generalizations of Discrete Groups (1999)
224. *M. Väth*, Volterra and Integral Equations of Vector Functions (2000)
225. *S. S. Miller and P. T. Mocanu*, Differential Subordinations (2000)
226. *R. Li et al.*, Generalized Difference Methods for Differential Equations: Numerical Analysis of Finite Volume Methods (2000)
227. *H. Li and F. Van Oystaeyen*, A Primer of Algebraic Geometry (2000)
228. *R. P. Agarwal*, Difference Equations and Inequalities: Theory, Methods, and Applications, Second Edition (2000)
229. *A. B. Kharazishvili*, Strange Functions in Real Analysis (2000)
230. *J. M. Appell et al.*, Partial Integral Operators and Integro-Differential Equations (2000)
231. *A. I. Prilepko et al.*, Methods for Solving Inverse Problems in Mathematical Physics (2000)

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

232. *F. Van Oystaeyen,* Algebraic Geometry for Associative Algebras (2000)
233. *D. L. Jagerman,* Difference Equations with Applications to Queues (2000)
234. *D. R. Hankerson et al.,* Coding Theory and Cryptography: The Essentials, Second Edition, Revised and Expanded (2000)
235. *S. Dăscălescu et al.,* Hopf Algebras: An Introduction (2001)
236. *R. Hagen et al.,* C*-Algebras and Numerical Analysis (2001)
237. *Y. Talpaert,* Differential Geometry: With Applications to Mechanics and Physics (2001)
238. *R. H. Villarreal,* Monomial Algebras (2001)
239. *A. N. Michel et al.,* Qualitative Theory of Dynamical Systems: Second Edition (2001)
240. *A. A. Samarskii,* The Theory of Difference Schemes (2001)
241. *J. Knopfmacher and W.-B. Zhang,* Number Theory Arising from Finite Fields (2001)
242. *S. Leader,* The Kurzweil-Henstock Integral and Its Differentials (2001)
243. *M. Biliotti et al.,* Foundations of Translation Planes (2001)
244. *A. N. Kochubei,* Pseudo-Differential Equations and Stochastics over Non-Archimedean Fields (2001)
245. *G. Sierksma,* Linear and Integer Programming: Second Edition (2002)
246. *A. A. Martynyuk,* Qualitative Methods in Nonlinear Dynamics: Novel Approaches to Liapunov's Matrix Functions (2002)
247. *B. G. Pachpatte,* Inequalities for Finite Difference Equations (2002)
248. *A. N. Michel and D. Liu,* Qualitative Analysis and Synthesis of Recurrent Neural Networks (2002)
249. *J. R. Weeks,* The Shape of Space: Second Edition (2002)
250. *M. M. Rao and Z. D. Ren,* Applications of Orlicz Spaces (2002)
251. *V. Lakshmikantham and D. Trigiante,* Theory of Difference Equations: Numerical Methods and Applications, Second Edition (2002)
252. *T. Albu,* Cogalois Theory (2003)
253. *A. Bezdek,* Discrete Geometry (2003)
254. *M. J. Corless and A. E. Frazho,* Linear Systems and Control: An Operator Perspective (2003)
255. *I. Graham and G. Kohr,* Geometric Function Theory in One and Higher Dimensions (2003)
256. *G. V. Demidenko and S. V. Uspenskii,* Partial Differential Equations and Systems Not Solvable with Respect to the Highest-Order Derivative (2003)
257. *A. Kelarev,* Graph Algebras and Automata (2003)
258. *A. H. Siddiqi,* Applied Functional Analysis (2004)
259. *F. W. Steutel and K. van Harn,* Infinite Divisibility of Probability Distributions on the Real Line (2004)
260. *G. S. Ladde and M. Sambandham,* Stochastic Versus Deterministic Systems of Differential Equations (2004)
261. *B. J. Gardner and R. Wiegandt,* Radical Theory of Rings (2004)
262. *J. Haluška,* The Mathematical Theory of Tone Systems (2004)

Additional Volumes in Preparation

E. Hansen and G. W. Walster, Global Optimization Using Interval Analysis: Second Edition, Revised and Expanded (2004)

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

PREFACE

The classical random flow and Newtonian mechanics approaches are the most extensively studied stochastic modeling methods for dynamic processes in biological, engineering, physical and social sciences. Both of these approaches lead to differential equations.

In the classical stochastic modeling approach, the state of a dynamic process is considered to be a random flow or process satisfying a certain probabilistic law such as Markov or diffusion. From these types of probabilistic assumptions, one then needs to determine the state transition probability distribution and density functions (STPDF). The determination of the unknown STPDF leads to the study of deterministic problems in the theory of ordinary, partial or integro-differential equations. These types of equations are referred to as master equations in the literature. The solution processes of such systems of differential equations are used to find the higher moments and other statistical properties of dynamic processes described by random flows.

On the other hand, the classical Newtonian mechanics type of stochastic modeling approach deals with a stochastic calculus to formulate stochastic mathematical models of dynamic processes. This approach leads directly to a system of stochastic differential equations, and its solution processes provide the description of the states of the dynamic processes as stochastic or random processes. This method of stochastic modeling generates three basic problems:

(i) Concepts of solution processes depending on modes of convergence and the fundamental properties of solutions: existence, uniqueness, measurability, continuous dependence on system parameters.

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

(ii) Probabilistic and statistical properties of solution process: probability distribution and density function, variance, and moments of solution processes and the qualitative/quantitative behavior of solutions.

(iii) Deterministic versus stochastic modeling of dynamic processes: If the deterministic mathematical model is available, then why do we need a stochastic mathematical model? If a stochastic mathematical model provides a better description of a dynamic process than the deterministic model, then the second question is to what extent the stochastic mathematical model differs from the corresponding deterministic model in the absence of random disturbances or fluctuations and uncertainties.

Most of the work on the theory of systems of stochastic differential equations is centered around problems (i) and (ii). This is because the theory of deterministic systems of differential equations provides many mathematical tools and ideas. It is problem (iii) that deserves more attention. Since 1970, some serious efforts have been made to address this issue in the context of stochastic modeling of dynamic processes by means of systems of stochastic differential equations. In the light of this interest, now is an appropriate time to present an account of stochastic versus deterministic issues in a systematic and unified way.

Two of the most powerful methods for studying systems of nonlinear differential equations are nonlinear variation of parameters and Lyapunov's second method. About a quarter century ago a hybrid of these two methods evolved. This hybrid method is called variational comparison method. In addition, a generalized variation of constants method has also developed in the same period of time. These new

techniques are very suitable and effective tools to investigate problems concerning stochastic systems of differential equations, in particular, stochastic versus deterministic issues.

This book offers a systematic and unified treatment for systems of stochastic differential equations in the framework of three methods: a) variational comparison method, b) generalized variation of constants method, and c) probability distribution method. The book is divided into five chapters. The first chapter deals with random algebraic polynomials. Chapter 2 is devoted to the initial value problem (IVP) for ordinary differential systems with random parameters. Stochastic boundary value problems (SBVP) with random parameters are treated in Chapter 3. Chapters 4 and 5 cover IVP and SBVP for systems of stochastic differential equations of Itô type, respectively.

A few important features of the monograph are as follows:

(i) This is the first book that offers a systematic study of the well-known problem of stochastic mathematical modeling in the context of systems of stochastic differential equations, namely, "stochastic versus deterministic;"

(ii) It complements the existing books in stochastic differential equations;

(iii) It provides a unified treatment of stability, relative stability and error estimate analysis;

(iv) It exhibits the role of randomness as well as rate functions in explicit form;

(v) It provides several illustrative analytic examples to demonstrate the scope of methods in stochastic analysis;

(vi) The methods developed in the book are applied to the existing stochastic mathematical models described by stochastic dif-

ferential equations in population dynamics, hydrodynamics, and physics;

(vii) Last but not least, it provides several numerical examples and figures to illustrate and compare the analytic techniques that are outlined in the book.

The monograph can be used as a textbook for graduate students. It can also be used as a reference book for both experimental and applied scientists working in the mathematical modeling of dynamic processes.

G. S. Ladde
M. Sambandham

CONTENTS

Chapter 3: Boundary Value Problems with Random Parameters

Chapter 4: Itô-Type Stochastic Differential Systems

Chapter 5: Boundary Value Problems of Itô-Type

APPENDIX

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

NOTATION AND ABBREVIATIONS

For the convenience of readers we list below the various notations and abbreviations employed in the monograph.

Vectors (column vectors) of dimension n are basically treated as $n \times 1$ matrices. All relations such as equations, inequalities, belonging to, and limits involving random variables or functions are valid with probability one. Sometimes the symbols $x(t)$ and $x(t, \omega)$ are used interchangeably as a random function.

R^n	As n-dimensional Euclidean space with a convenient norm $\|\bullet\|$
$\|\bullet\|$	The norm of a vector or matrix
R	The set of all deterministic real numbers or a real line
R_+	The set of all $t \in R$ such that $t \geq 0$
I	An arbitrary index set, in particular, a finite, countable set, or any interval in R
$I(1, n)$	$\{1, 2, \ldots, n\}$, that is, the set of first n positive integers
J	$[t_0, t_0 + a]$, where $t_0 \in R$ and a is a positive real number
$B(z, \rho)$	The set of all $x \in R^n$ such that $\|x - z\| < \rho$ for given $z \in R^n$ and positive real number ρ
$\overline{B}(z, \rho)$	The closure of $B(z, \rho)$
$B(\rho)$	The set $B(z, \rho)$ with $z = 0 \in R^n$
$\mathcal{F}^n$	The σ-algebra of Borel sets in R^n
$\mathcal{B}$	The σ-algebra of Borel sets in a metric space (X, d), where d is a metric induced by the norm $\|\bullet\|$ and X is a separable Banach space

$(\Omega, \mathcal{F}, P) \equiv \Omega$ A complete probability space, where Ω is a sample space, $\mathcal{F}$ is a σ-algebra of Ω, and P is a probability measure defined on $\mathcal{F}$

$\mathcal{F}_t$ A sub-σ-algebra of $\mathcal{F}$ for $t \in R_+$

$\mathcal{L}_t$ The smallest sub-σ-algebra of $\mathcal{F}$ generated by a k-dimensional normalized Wiener process $z(t)$ for $t \in R_+$

$R[\Omega, R^n]$ The collection of all random vectors defined on complete probability space $(\Omega, \mathcal{F}, P)$ into R^n

$R[\Omega, R^{nm}] \equiv [\Omega, \mathcal{M}_{n\times m}]$ A collection of all $n \times m$ random matrices $A(\omega) = (a_{ij}(\omega))$ such that $a_{ij} \in R[\Omega, R]$

$\|x\|_p$ $\|x\|_p = \Big(E[\|x(\omega)\|^p]\Big)^{1/p} = \Big(\int_\Omega \|x(\omega)\|^p P(d\omega)\Big)^{1/p}$ for $p \geq 1$

$\mathcal{L}^p$ The collection of all n-dimensional random vectors x such that $E[\|x(\omega)\|^p] < \infty$ for $p \geq 1$

$L^p[\Omega, R^n]$ A collection of all equivalence classes of n-dimensional random vectors such that an element of an equivalence class belongs to $\mathcal{L}^p$

$R[[a,b], R[\Omega, R^n]] \equiv R[[a,b] \times \Omega, R^n]$ A collection of all R^n-valued separable random functions defined on $[a,b]$ with a state space $(R^n, \mathcal{F}^n)$, $a, b \in R$

$M[[a,b], R[\Omega, R^n]] \equiv M[[a,b] \times \Omega, R^n]$ A collection of all random functions in $R[[a,b], R[\Omega, R^n]]$ which are product-measurable on $([a,b] \times \Omega, \mathcal{F}^1 \times \mathcal{F}, m \times P)$, where $(\Omega, \mathcal{F}, P) \equiv \Omega$ and $([a,b], \mathcal{F}^1, m)$ are complete probability and Lebesgue-measurable spaces, respectively

$M[R_+ \times R^n, R[\Omega, R^n]] \equiv M[R_+ \times R^n \times \Omega, R^n]$ A class of R^n-valued

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

random functions $F(t, x, \omega)$ such that $F(t, x(t, \omega), \omega)$ is product measurable whenever $x(t, \omega)$ product is measurable

$M[[0,1] \times R^n \times R^n, R[\Omega, R^n]] \equiv M[[0,1] \times R^n \times R^n \times \Omega, R^n]$ A class of R^n-valued random functions $F(t, x, y, \omega)$ such that $F(t, x(t, \omega), y(t, \omega), \omega)$ is product measurable whenever $x(t, \omega)$ and $y(t, \omega)$ are product measurables

$C[D, R^n]$ The class of all deterministic continuous functions defined on an open (t, x) subset D of R^{n+1} into R^n

$C[R_+ \times R^n, R^m]$ The class of all deterministic continuous functions defined on $R_+ \times R^n$ into R^m

$C[[0,1] \times R^n \times R^n, R^m]$ The collection of all deterministic continuous functions $[0,1] \times R^n \times R^n$ into R^m

$C[[a,b], R[\Omega, R^n]] \equiv C[R_+ \times R^n, R[\Omega, R^n]]$ A collection of all sample continuous R^n-valued random functions $x(t, \omega)$

$C[R_+ \times R^n, R[\Omega, R^n]] \equiv C[R_+ \times R^n \times \Omega, R^n]$ A class of sample continuous R^n-valued random functions $F(t, x, \omega)$ defined on $R_+ \times R^n \times \Omega$ into R^n

$C[[0,1] \times R^n \times R^n, R[\Omega, R^n]] \equiv C[[0,1] \times R^n \times R^n \times \Omega, \mathcal{R}^n]$ A class of sample continuous R^n-valued random functions $F(t, x, y, \omega)$ defined on $[0,1] \times R^n \times R^n \times \Omega$ into R^n

A^T The transpose of a vector or matrix A

A^{-1} The inverse of a square matrix A

$tr(A)$ The trace of a square matrix A

$\det(A)$ The determinant of a square matrix A

$\mu(A(\omega))$	The logarithmic norm of a random square matrix $A(\omega)$
$\sigma(A(\omega))$	The spectrum of a random square matrix $A(\omega)$
a.e.	Almost everywhere or except a set of measure zero
a.s.	Almost surely or almost certainly
i.p.	In probability or stochastically
m.s.	Mean square or quadratic mean
p.m.	pth mean or moment
w.p. 1	With probability one
DIVP	Deterministic Initial Value Problem
SIVP	Stochastic Initial Value Problem
RIVP	Random Initial Value Problem
DBVP	Deterministic Boundary Value Problem
SBVP	Stochastic Boundary Value Problem
$I_A \equiv \chi_A$	A characteristic or indicator function with respect to an event A
$\mathcal{K}$	The class of functions $b \in C[[0,\rho), R_+]$ such that $b(0) = 0$ and $b(r)$ is strictly increasing in r, where $0 \leq \rho < \infty$
$\mathcal{VK}$	The class of functions $b \in C[[0,\rho), R_+]$ such that $b(0) = 0$ and $b(r)$ is convex and strictly increasing in r, where $0 \leq \rho < \infty$
$\mathcal{CK}$	The class of functions $a \in C[R_+ \times [0,\rho), R]$ such that $a(t,0) \equiv 0$ and $a(t,r)$ is concave and strictly increasing in r for each $t \subset R_+$, where $0 \leq \rho < \infty$
$\mathcal{L}$	The class of functions $\alpha \in C[[0,\infty), R_+]$ such that $\alpha(0) = 0$ and $\alpha(r) \to 0$ as $r \to \infty$
$E[x]$	The expected value of a random variable x
$\mathrm{Var}[x]$	The variance of a random variable x

$V_x(t,x) \equiv \frac{\partial V}{\partial x}(t,x)$	An $m \times n$ Jacobial matrix of $V(t,x)$, where $V \in C[R_+ \times R^n, R^m]$, V_x exists
$V_{xx}(t,x) \equiv \frac{\partial^2 V}{\partial x^2}(t,x)$	An $n \times n$ Hessian matrix of $V(t,x)$ of $V \in C[R_+ \times R^n, R^m]$ whose elements $\frac{\partial^2 V}{\partial x_i \partial x_j}(t,x)$ are m-dimensional vectors
$\|x(\omega)\|_0$	$\max \|x(t,\omega)\|$, for $x \in L^\infty[[0,1], R[\Omega, R^n]]$ $t \in J$

CHAPTER 1: RANDOM POLYNOMIALS

1.0. INTRODUCTION

An estimate on the variation of a random function with the corresponding smooth function is presented in Section 1.1. Section 1.2 deals with the absolute mean deviation of solutions of random polynomial equations with solutions of the corresponding polynomial equations whose coefficients are the mean of the coefficients of the random polynomials. A brief account about the eigenvalues of random matrices and its stability are presented in Sections 1.3 and 1.4, respectively. In Section 1.5, the mathematical results are applied to the problems in (i) economic analysis of capital and investment, and (ii) free damped motion of a spring. In addition, several numerical examples are presented in Section 1.6 illustrating the scope and the usefulness of the mathematical theory.

1.1. UPPER BOUND FOR MEAN DEVIATION

In this section, we develop a result which estimates the difference between a random function and the corresponding smooth function

Lemma 1.1.1. *Let $f \in M[R^{n+1}, R]$ and let $x \in R[\Omega, R^{n+1}]$. Further assume that $f(x)$ be analytic on a poly disc*

$$H_R = \{z \in C^{n+1} : |z_i - E(x_i(\omega))| < R, i = 0, 1, 2, \ldots, n\}$$

of radius R centered at $E(x(\omega))$ and let $0 < \Delta < R$. Then

$$|f(x(\omega)) - f\big(E(x(\omega))\big)| \le L(\omega) + M \sum_{k=1} \chi_{\Delta} t^{|k|}, \tag{1.1.1}$$

where

$$|\chi_{\Delta'} f(x(\omega))| \le L(\omega),$$

$$M = \sup\{|f(x(\omega))| : x(\omega) \in \partial H_R\},$$

$$t = \frac{\|x(\omega) - E(x(\omega))\|}{R}$$

and

$$|k| = \sum_{i=0}^{n} k_i.$$

Proof. Given that $f(x(\omega))$ is analytic on H_R centered at $E(x(\omega))$ and $0 < \Delta < R$. Let us denote by χ_Δ and $\chi_{\Delta'}$ the indicator functions of H_Δ and $H_{\Delta'}$, respectively, for $0 < \Delta < R$, where

$$H_\Delta = \{z \in C^{n+1} : |z_i - E(x_i(\omega))| < \Delta, i = 0, 1, ..., n\}, \qquad (1.1.2)$$

and $H_{\Delta'}$ is complement of H_Δ. Then

$$f(x(\omega)) = \chi_{\Delta'} f(x(\omega)) + \chi_\Delta f(x(\omega)). \qquad (1.1.3)$$

Assume that

$$|\chi_{\Delta'} f(x(\omega))| \leq L(\omega), \quad \text{a.s.} \qquad (1.1.4)$$

for some random variable $L(\omega)$. Since $H_\Delta \subset H_R$, $f(z)$ is still analytic on H_Δ and hence by Taylor series

$$\chi_\Delta f(z) = \sum_{k=0} \frac{1}{k!} D^k f\big(E(x(\omega))\big)(\chi_\Delta y^k), \qquad (1.1.5)$$

where

$$y_i = z_i - E(x_i(\omega)),$$

$$D^k = \frac{\partial^{|k|}}{\partial^{k_0} x_0 \partial^{k_1} x_1 \cdots \partial^{k_n} x_n}, \qquad k = (k_0, k_1, \ldots, k_n),$$

$$|k| = \sum_{i=0}^{n} k_i, \qquad k! = k_0! k_1! \cdots k_n! \quad \text{and } x^k = x_0^{k_0} x_1^{k_1} \cdots x_n^{k_n}.$$

(1.1.5) can be rewritten as follows

$$\chi_\Delta f(x(\omega)) = f(E(x(\omega))) + \sum_{k=1} \frac{1}{k!} D^k f\big(E(x(\omega))\big)(\chi_\Delta \delta x(\omega)^k), \tag{1.1.6}$$

where $\delta x(\omega) = x(\omega) - Ex(\omega)$. Since

$$|D^k f| \le Mk! R^{-|k|},$$

$$M = \sup\{|f(z)| : z \in \partial H_R\},$$

we can rewrite the second term in right hand side of (1.1.6) as follows

$$\left| \sum_{k=1} \frac{1}{k!} D^k f(E(x(\omega)))(\chi_\Delta \delta x(\omega)^k) \right| \le M \sum_{k=1} \chi_\Delta \left(\frac{\|\delta x(\omega)\|^k}{R} \right). \tag{1.1.7}$$

From (1.1.3), (1.1.4), (1.1.6) and (1.1.7), we get

$$\|f(x(\omega)) - f(E(x(\omega)))\| \le L(\omega) + M \sum_{k=1} \chi_\Delta \left(\frac{\|\delta x(\omega)\|}{R} \right)^k. \tag{1.1.8}$$

By denoting $\|\delta x(\omega)\|/R$ by t we establish the proof of the lemma.

We illustrate the usefulness of this lemma in the following example.

Example 1.1.1. Let $p(a(\omega)) = a_0 + a_1 x + \cdots + a_n x^n$ and $\xi_i(a)$, $i = 1, 2 \ldots, n$ be the roots of $p(a(\omega))$. Let $p(E(a)) = \widehat{a}_0 + \widehat{a}_1 x + \cdots + \widehat{a}_n x^n$ and $\xi_i(E(a))$, $i = 1, 2, \ldots, n$ be the roots of $p(E(a))$. Suppose that all the hypothesis of Lemma 1.1.1 with $f(z) = \xi_i(z)$ are satisfied. By an application of Cauchy estimate we get

$$|\xi_i(a)| \le 1 + \max_{0 \le i \le n} |a_i|. \tag{1.1.9}$$

By an application of Krengel and Sucheston Theorem A.1.1 we get

$$E\left\{ \chi_{\Delta'} \max_{0 \le i \le n} |a_i| \right\} \le 2 \max_{0 \le i \le n} E\{\chi_{\Delta'} |a_i|\}. \tag{1.1.10}$$

Taking expectation on both the sides of (1.1.8) and by an application of (1.1.10) we obtain

$$E|\xi_i(a) - \xi_i(E(a))| \leq 2 \max_{0 \leq i \leq n} +M \sum_{k=1} E(\chi_\Delta t^k), \tag{1.1.11}$$

where

$$M = 1 + \max_{0 \leq i \leq n} \{R + |Ea_i|\},$$

$$t = \frac{\|a - E(a)\|}{R}.$$

1.2. ERROR ESTIMATES

In this section we develop an error estimate which is more efficient for random polynomial. Let $a_i(\omega)$, $i = 0, 1, \ldots, n$ be complex valued random variables defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with $a_n(\omega) \neq 0$ with probability one. In this section we consider some mathematical results concerning algebraic polynomial

$$P_n(z, \omega) = a_0(\omega) + a_1(\omega)z + \cdots + a_n(\omega)z^n, \tag{1.2.1}$$

where $z \in Z$, and Z is set of complex numbers. The polynomial corresponding to the mean of the coefficients of (1.2.1) is

$$\begin{aligned} \widehat{P}_n(z) &= E(P_n(z, \omega)) \\ &= E(a_0(\omega)) + E(a_1(\omega))z + \cdots + E(a_n(\omega))z^n. \end{aligned} \tag{1.2.2}$$

Any $\bar{z}_0 \in Z$ that satisfies $P_n(\bar{z}_0, \omega) = 0$ is called a zero of (1.2.1) or solution of the random algebraic equation $P_n(z, \omega) = 0$. From Theorem A.1.2, one can conclude that under general conditions and for sufficiently large n, the distribution of zeros of the random algebraic polynomial $P_n(z, \omega)$ is, in a certain sense, close to the uniform distribution on the circumference of the unit circle with center at the

origin. In the same way we can ask that where exactly the sample roots cluster as the sample size increases. We answer this in the following.

If N independent samples of a are taken, say, $a^{(j)}$, $j = 1, 2, \ldots, N$, we see that the values of the roots $\xi_i^j = \xi_i(a^{(j)})$, where $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, N$ are independent for fixed i and any j. Also for fixed i, they will be independently distributed with

$$\begin{aligned} E|\xi_i^j| = E|\xi_i^j| &\le E\left[1 + |a_0^{(1)}| + \cdots + |a_{n-1}^{(1)}|\right] \\ &= 1 + \sum_{i=0}^{n-1} E\left[|a_i^{(1)}|\right] \le 1 + nE\left[|a_{n-1}^{(1)}|\right] < \infty, \end{aligned} \tag{1.2.3}$$

and

$$\operatorname{Var}\left[\xi_i^j\right] = \operatorname{Var}\left[\xi_i^1\right] \le E\left[|\xi_i^1|\right] \le E\left[1 + \sum_{i=0}^{n-1} |a_i^{(1)}|^2\right] < \infty. \tag{1.2.4}$$

Here we have used the estimate $|\xi_i^1| \le 1 + \sum_{i=0}^{n-1} |a_i^1|$, which holds for $a_n = 1$. In the light of (1.2.3), (1.2.4) and applying the strong law of large numbers (Theorem A.1.8), we obtain

$$\sum_{j=1}^{N} \frac{\xi_k^j}{N} \to E\left[\xi_k^1\right], \text{ w.p. } 1. \tag{1.2.5}$$

Also by the Lindeberg-Feller Theorem A.1.7, it follows that

$$\sum_{j=1}^{N} \frac{(\xi_k^j - E[\xi_k^j])}{\sqrt{N \operatorname{Var}[\xi_k^1]}} \to N(0,1), \tag{1.2.6}$$

in distribution for every fixed k. This completes the proof of the lemma.

In view of (1.2.5) and (1.2.6) we would like to determine $E\left[\xi_i^1(a)\right]$ and compare it with $\xi_i^1(E[a])$, the i^{th} root of (1.2.2), that is, we want to estimate $|E\left[\xi_i^1(a)\right] - \xi_i^1(E[a])|$. For the sake of simplicity, we denote $\xi_i^1(a)$ by $\xi(a)$.

Theorem 1.2.1. *Let $\xi(a)$ be a root function for the random algebraic polynomial (1.2.1) with $a_n(\omega) \equiv 1$ w.p. 1. Assume that a_i, $i = 0, 1, \ldots, n-1$, are independent, identically distributed and symmetric about their means. Assume that ξ is analytic on H_R, the poly disc of radius R centered at $E[a]$ and let $0 < \Delta < R$. Then*

$$|E[\xi(a)] - \xi(E[a])| \leq ER(\Delta), \tag{1.2.7}$$

where

$$ER(\Delta) = 2 \max_{0 \leq i \leq n-1} E\left[\chi'_\Delta |a_i|\right] + E\left[\chi_\Delta\right] |1 - \xi(E[a])| + L(I_n^n - I_3^n),$$

$$L = 1 + \max_{0 \leq i \leq n-1} [R + |E[a_i]|],$$

$$I_2 = E\left[\frac{\chi_\Delta}{(1-t^2)}\right],$$

$$I_3 = E\left[\chi_\Delta\right],$$

$$t = \frac{|z_1 - E\left[a_1\right]|}{R}.$$

Proof. We prove this theorem by means of Taylor expansion of ξ on a suitably large poly disc centered at $E[a]$, that is

$$H_R = \{z \in C^n : |z_i - E\left[a_i\right]| < R, i = 0, 1, \ldots, n-1\}$$

and let

$$H_\Delta = \{z \in C^n : |z_i - E\left[a_i\right]| < \Delta, i = 0, 1, ..., n-1\},$$

χ_Δ and $\chi_{\Delta'}$ are indicator functions of H_Δ and its complement $H_{\Delta'}$, for $0 < \Delta < R$. Now we can write

$$E[\xi(a)] = E\left[\chi_{\Delta'}\xi(a)\right] + E\left[\chi_\Delta \xi(a)\right]. \tag{1.2.8}$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

Since $H_\Delta \subset H_R$, ξ will be analytic on H_Δ and hence the Taylor series may be used in the estimation of the second term in (1.2.8). Hence

$$\begin{aligned} E\{\chi_\Delta \xi\} &= \sum_{k=0} \frac{1}{k!} D^k \xi(E[a]) E[\chi_\Delta x^k] \\ &= \xi(E[a]) E[\chi_\Delta] + \Sigma' \frac{1}{k!} D^k \xi(E[a]) E\left[\chi_\Delta x^k\right] \\ &\quad + \Sigma'' \frac{1}{k!} D^k \xi(E[a]) E\left[\chi_\Delta x^k\right]. \end{aligned} \tag{1.2.9}$$

Here in Σ' at least one of the k_i is odd and in Σ'' all are even and at least one k_i is greater than zero. Let $x_i = z_i - E[a_i]$. This implies that x_i are independent and symmetric. Therefore $E\left[\chi_\Delta x_i^{k_i}\right] = 0$ for all odd k_i and Σ' term vanishes. Therefore (1.2.9) reduces to

$$E[\chi_\Delta \xi] = \xi(E[a]) E[\chi_\Delta] + \Sigma'' \frac{1}{k!} D^k \xi \prod_{i=0}^{n-1} E\left[\chi_\Delta x_i^{k_i}\right]. \tag{1.2.10}$$

Let $M = \sup\{|\xi(a)| : z \in \partial H_R\}$. By an application of the Cauchy estimate, namely,

$$|D^k \xi| \leq M k! R^{-|k|}, \tag{1.2.11}$$

we obtain

$$|I_1| = |\Sigma'' \frac{1}{k!} D^k \xi \prod_{i=0}^{n-1} E\left[\xi_\Delta x_i^{k_i}\right]| \leq M \Sigma'' \prod_{i=0}^{n-1} E\left[\chi_\Delta (|x_i|/R)^{k_i}\right]. \tag{1.2.12}$$

Let h denote the number of k_i equal to zero in each term of Σ''. Because of the fact that x_i are identically distributed, we get by induction on n that

$$|I_1| \leq M \sum_{h=0}^{n-1} \binom{n}{h} (E[\chi_\Delta])^h \left(\sum_{j=1}^{\infty} E\left[\chi_\Delta \left(\frac{|x_1|^{2j}}{R}\right)\right]\right)^{n-h}. \tag{1.2.13}$$

Setting $t = |x_1|/R$, we define

$$I_2 = \sum_{j=0}^{\infty} E[\chi_\Delta t^{2j}] = E\left[\frac{\chi_\Delta}{1-t^2}\right], \tag{1.2.14}$$

$$I_3 = E[\chi_\Delta].$$

Therefore from (1.2.14), (1.2.13) can be rewritten as

$$|I_1| \leq M[I_2^n - I_3^n]. \tag{1.2.15}$$

From (1.2.8), (1.2.9), (1.2.10), (1.2.11), (1.2.15), and noticing the fact that $M \leq 1 + \max_{0 \leq i \leq n-1}\{R + |E[a_i]|\}$, we obtain (1.2.7). This completes the proof of the theorem.

Remark 1.2.1. Theorem 1.2.1 gives the error estimate between the mean root of the random equation and the root of the mean equation. Further it shows that as the sample size increases the samples corresponding to a given root cluster about its average.

In the following corollaries the error $E(R(\Delta))$ in Theorem 1.2.1 is improved by removing all terms of specific order from the summation (1.2.9). Doing this for $|k| = 2$ we get the following corollary.

Corollary 1.2.1. *Suppose that all the hypothesis of* Theorem 1.2.1 *hold. Further assume that*

$$m_2 = \max_{0 \leq i \leq n-1} \left\{ \left| \frac{\partial^2 \xi}{\partial a_i^2} \, E\,[a] \right| \right\} < \infty.$$

Then

$$\begin{aligned} |E\,[\xi(a)] - \xi\,(E[a])\,| &\leq ER_2(\Delta) \\ &= E[R(\Delta)] + n\left(\frac{m_2}{2} - L\right) E^{n-1}\,[\chi_\Delta]\,E[\chi_\Delta t^2]. \end{aligned}$$

Likewise, for $|k| = 4$ we get the following corollary.

Corollary 1.2.2. *Let all the hypothesis of* Theorem 1.2.1 *be satisfied. Suppose that*

$$m_4 = \max_{0 \leq i,j \leq n-1} \left\{ \left| \frac{\partial^4 \xi}{\partial a_i^2 \partial a_j^2} E[a] \right| \right\} < \infty.$$

Then

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

$$
\begin{aligned}
|E\left[\xi(a)\right] - \xi\left(E[a]\right)| \leq ER_4(\Delta) = ER_2(\Delta) \\
&+ n\left(\frac{m_4}{4} - L\right) E^{n-1}\left[\chi_\Delta\right] E[\chi_\Delta t^4] \\
&+ \frac{n(n-1)}{2}\left(\frac{m_4}{2!2!} - L\right) E^{n-2}[\chi_4] E^2[\chi_\Delta t^2].
\end{aligned}
$$

Suppose now that $a = (a_0, a_1, \ldots, a_n)$ be dependent random variables such that $E\left[|a|\right] < \infty$ and $\mathrm{Var}[a] < \infty$. If N independent samples are taken, say, $a^{(j)}$, $j = 1, 2, \ldots, N$, we see that the value of the roots $\xi_i^j = \xi(a^j)$, for $i = 1, 2, \ldots, n$, $j = 1, 2, ..., N$ will be independent for fixed i and variable j and dependent for fixed j and variable i. We note that (1.2.3)–(1.2.6) are satisfied. The next theorem deals with coefficients of random polynomial whose coefficients are dependent random variables.

Theorem 1.2.2. *Let a_i, $i = 0, 1, \ldots, n-1$, $a_n \equiv 1$ be dependent random variables satisfying* (1.2.3)–(1.2.6). *Assume that $\xi(a)$ is defined on H_R, the poly disc centered at $E(a)$ with $0 < \Delta < R$. Then*

$$|E[\xi(a)] - \xi\left(E[a]\right)| < ER(\Delta),$$

where

$$
\begin{aligned}
ER(\Delta) = E\left[\chi_{\Delta'} \max_{0 \leq i \leq n-1} |a_i|\right] + E\left[\chi_{\Delta'}\right] |1 - \xi(E(a))| \\
+ M \sum_{k=1} E\left[\chi_\Delta \left(\frac{|x|}{R}\right)^k\right], \quad (1.2.16)
\end{aligned}
$$

$$M \leq 1 + \max_{0 \leq i \leq n-1} (R + E[|a_i|]),$$

$$|x| = |(z_0 - E[a_0])(z_1 - E[a_1]) \cdots (z_{n-1} - E\left[a_{n-1}\right])|.$$

Proof. Since $H_\Delta \subset H_R$ and $\xi(z)$ is analytic, applying Taylor series

we have

$$E\left[\chi_\Delta \xi(a)\right] = \sum_{k=0} \frac{1}{k!} D^k \xi(E[a]) E\left[\chi_\Delta x^k\right]$$
$$= \xi(E[a]) E\left[\chi_\Delta\right] + \sum_{k=1} \frac{1}{k!} \xi\left(E[a]\right) E\left[\chi_\Delta x^k\right]. \tag{1.2.17}$$

Using Cauchy estimates (1.2.11) we get

$$E\left[\chi_\Delta \xi(a)\right] \leq \xi\left(E[a]\right) E[\chi_\Delta] + M\Sigma E\left[\chi_\Delta \left(\frac{|x|}{R}\right)^k\right]. \tag{1.2.18}$$

From (1.2.7), (1.2.8), (1.2.15) and (1.2.16), we obtain

$$|E[\xi(a)] - \xi(E[a])| \leq E\left[\chi_{\Delta'} \max_i |a_i|\right]$$
$$+ E\left[\chi_{\Delta'}\right] |1 - \xi(E[a])| + M\Sigma E\left[\chi_\Delta \left(\frac{|x|}{R}\right)^k\right], \tag{1.2.19}$$

where $|x| = |(z_0 - E[a_0])(z_1 - E[a_1]) \cdots (z_{n-1} - E[a_{n-1}])|$. This completes the proof of the theorem.

Example 1.2.1. This example illustrates the difference between mean of the roots and roots of the mean equation. Let $n = 19$, a_0 is from $N(-1, 1)$, $a_1, a_2, \ldots, a_{n-1}$ are from $N(0, 1)$ and $a_n = 1$. The mean of the random algebraic polynomial is $x^n - 1 = 0$. Let the correlation between any two random variables be 0.5. We used IMSL subroutine to obtain the random numbers and to solve random polynomials. We used GGNML to obtain random numbers and ZPPOLY to obtain solution of random polynomials. For details, please, see the numerical example section.

1.3. EIGENVALUES OF RANDOM MATRICES

In this section, we present a few results on random matrices and its related characteristic polynomials. In addition, the notion of logarithmic norm for random matrices and its relation with eigenvalues are also discussed.

Let $A(\omega) = (a_{ij}(\omega))_{n\times n}$ be $n \times n$ random matrix, whose elements $a_{ij}(\omega)$ are random variables defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with values in a set of real number. An $n \times n$ random matrix $A \in [\Omega, \mathcal{M}_{n\times n}]$, where $\mathcal{M}_{n\times n}$ is the Banach Algebra of $n \times n$ matrices, that is, A is an $n \times n$ matrix-valued random variable.

For many applications, one basic problem is the following: given an $(n \times n)$ random matrix, find a n-dimensional random vector $c(\omega) = c(A(\omega))$ and a random number $\lambda(\omega) = \lambda(A(\omega))$ (real/complex) such that

$$A(\omega)c(\omega) = \lambda(\omega)c(\omega). \tag{1.3.1}$$

This condition is equivalent to solving the following linear system of random algebraic equations

$$A(\omega)c - \lambda c = 0$$

or

$$\big(A(\omega) - \lambda I\big)c = 0, \tag{1.3.2}$$

where $I = I_{n\times n}$ is identity marix. For the existence of a non-zero random vector $c(\omega)$, it is necessary and sufficient that

$$\det\big(A(\omega) - \lambda I\big) = 0, \tag{1.3.3}$$

where "det" stands for the determinant of the random matrix $\big(A(\omega) - \lambda I\big)$. Equation (1.3.3) is referred as the characteristic equation of the random matrix $A(\omega)$. In fact, it is a n-th degree polynomial equation in λ with random coefficients that are random functions of elements $a_{ij}(\omega)$ of $A(\omega)$. Moreover, its leading coefficient is always 1. Hence the characteristic equation (1.3.3) always has n roots (real or complex) which are not necessarily all distinct. For each one of

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

these random characteristic roots $\lambda_k(A(\omega))$, there exists at least one random vector $c_k(\omega) = c_k(A(\omega))$ satisfying (1.3.1). Traditionally, $1/\lambda_k(A(\omega))$ is called an eigenvalue of the random matrix $A(\omega)$. But usually, $\lambda_k(A(\omega))$ is called an eigenvalue of the random matrix $A(\omega)$ and $c_k(\omega)$ is the corresponding eigenvector. For the details about the theory of random determinants, see [118, 8]. The results developed in the preceeding sections can be applied to the above types of characteristic polynomial equations.

Let $\widehat{A}$ be an $n \times n$ matrix that corresponds to the random matrix $A(\omega)$. Then eigenvalues and eigenvectors that corresponds to this smooth matrix $\widehat{A}$ are determined by

$$\widehat{A}m - \lambda m = 0. \tag{1.3.4}$$

Again, for the existence of a non-zero random vector m, it is necessary and sufficient that

$$\det\big(\widehat{A} - \lambda I\big) = 0. \tag{1.3.5}$$

This is a n-th degree polynomial equation in λ with deterministic coefficients with the same feature as described above. An easy candidate for $\widehat{A}$ is $E[A(\omega)]$.

In many applications, it is enough to know an estimate on the real part of the eigenvalues of a random matrix. This is because of the fact that the task of finding eigenvalues of a random matrix is very difficult, particularly, if the size $(n \times n)$ of the random matrix is very very large. The logarithmic norm of a random matrix $A(\omega)$ is algebraically simple, easy to verify, and computationally attractive.

Let $\|\cdot\|$ stand for the norm of n-dimensional vector or $n \times n$ matrix. We define the following random function:

$$a(t,\omega) = \frac{1}{t}\big[\big\|I + tA(\omega)\big\| - 1\big], \tag{1.3.6}$$

where $\|\cdot\|$ is a matrix norm, I is as defined before; t is a positive real number.

Now we prove the following result:

Lemma 1.3.1. *The random function $a(t,\omega)$ defined in (1.3.6) is sample continuous in $t > 0$.*

Proof. For any fixed $t_0 > 0$, and any $t > 0$, $\omega \in \Omega$, consider

$$\begin{aligned}|a(t,\omega) - a(t_0,\omega)| &= \Big|\frac{1}{t}\big[\|I + tA(\omega)\| - 1\big] - \frac{1}{t_0}\big[\|I + t_0A(\omega)\| - 1\big]\Big| \\ &\leq \frac{1}{tt_0}\Big[\big|\big[t_0\|I + tA(\omega)\| - t\|I + t_0A(\omega)\|\big]\big| + |t - t_0|\Big] \\ &\leq \frac{2}{tt_0}\,|t - t_0|.\end{aligned}$$

This shows that

$$P\{\omega : \lim_{t\to t_0} |a(t,\omega) - a(t_0,\omega)| = 0\} = 1.$$

Remark 1.3.1. From Lemma 1.3.1 and by the application of Theorem 2.6 of Doob [31], we can conclude that the stochastic process $a(t,\omega)$ is separable.

Now we are in position to define the concept of logarithmic norm for a random matrix $A(\omega)$.

Definition 1.3.1. Let $A(\omega)$ be an $n \times n$ random matrix defined on a complete probability space $(\Omega, \mathcal{F}, P)$. If the following limit

$$\lim_{t\to 0^+} \frac{1}{h}\big[\|I + hA(\omega)\| - 1\big], \quad \text{w.p. } 1 \tag{1.3.7}$$

exists, then the limit is called the logarithmic norm of $A(\omega)$, and it is denoted by $\mu(A(\omega))$.

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

Remark 1.3.2. From Remark 1.3.1, it is obvious that the logarithmic norm $\mu(A(\omega))$ of the $n \times n$ random matrix $A(\omega)$ is a random variable.

The following result provides an estimte on the real part of any eigenvalue of a given random matrix $A(\omega)$.

Lemma 1.3.2. *Let $\sigma(A(\omega))$ be the spectrum of $n \times n$ random matrix $A(\omega)$. Then*

$$\mathrm{Re}\lambda(A(\omega)) \leq \mu(A(\omega)), \quad \textit{for all } \lambda(A(\omega)) \in \sigma(A(\omega)). \tag{1.3.8}$$

Proof. For any $\lambda(A(\omega)) \in \sigma(A(\omega))$, and $c(\omega)$ is a corresponding characteristic vector with norm 1. Then, for an arbitrary $h > 0$,

$$\begin{aligned} \big\|\big(I + hA(\omega)\big)c(\omega)\big\| - \big\|c(\omega)\big\| &= \big|1 + h\lambda(A(\omega))\big|\big\|c(\omega)\big\| - \big\|c(\omega)\big\| \\ &= \big|1 + h\lambda(A(\omega))\big| - 1 \\ &= h \ \mathrm{Re}\ \lambda(A(\omega)) + o(h). \end{aligned} \tag{1.3.9}$$

On the other hand, from (1.3.7), we have

$$\big\|\big(I + hA(\omega)\big)c(\omega)\big\| - \big\|c(\omega)\big\| \leq \big\|\big(I + hA(\omega)\big)\big\| - 1 \leq h\mu(A(\omega)) + o(h).$$

This together with (1.3.9), yields

$$\mathrm{Re}\ \lambda(A(\omega)) \leq \mu(A(\omega)), \quad \text{for all } \lambda(A(\omega)) \in \sigma(A(\omega)).$$

This completes the proof of the lemma.

Remark 1.3.3. Further additional properties of logarithmic norm for random matrix functions are outlined in Lemma A.2.3. These properties are natural extensions of the properties of deterministic

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

matrix functions. In fact, the following property is of special interest in the context of the main objective of this book:

$$-\|A(\omega) - \widehat{A}\| \le \mu\big(A(\omega) - \widehat{A}\big) \le \|A(\omega) - A\|. \tag{1.3.10}$$

In the following we give an example which shows the logarithmic norm of a matrix which depends on the norm of the matrix.

Example 1.3.1. For an $n \times n$ random marix $A(\omega)$, we consider the following norms:

(MN_1) $\|A(\omega)\|_E =$ square root of the largest eigenvalue of $\big(A^T(\omega)A(\omega)\big)$;

(MN_2) $|A(\omega)\|_{1R} = \sup_{i\in I(1,n)}\big[\sum_{k=1}^n |a_{ik}(\omega)|\big]$;

(MN_3) $\|A(\omega)\|_{1C} = \sup_{k\in I(1,n)}\big[\sum_{i=1}^n |a_{ik}(\omega)|\big]$;

(MN_4) $\|A(\omega)\|_{dC} = \sup_{k\in I(1,n)}\big[\sum_{i=1}^n d_k^{-1} d_i |a_{ik}(\omega)|\big]$;

(MN_5) $\|A(\omega)\|_Q =$ square root of the largest eigenvalue of $\big(Q^{-1}A^T(\omega)QA(\omega)\big)$ for some positive definite matrix Q.

Then, from the definition of $\mu(A(\omega))$, it is easy to see that

(i) (MN_1) implies $\mu(A(\omega))$ = the largest eigenvalue of $\frac{1}{2}[A^T(\omega) + A(\omega)]$.

(ii) (MN_2) implies $\mu(A(\omega)) = \sup_{i\in I(1,n)}\big[a_{ii}(\omega) + \sum_{\substack{k=1\\k\neq i}}^n |a_{ik}(\omega)|\big]$.

(iii) (MN_3) implies $\mu(A(\omega)) = \sup_{k\in I(1,n)}\big[a_{kk}(\omega) + \sum_{\substack{i=1\\i\neq k}}^n |a_{ik}(\omega)|\big]$.

(iv) (MN_4) implies $\mu(A(\omega)) = \sup_{k\in I(1,n)}\big[a_{kk}(\omega) + \sum_{\substack{i=1\\i\neq k}}^n d_k^{-1} d_i |a_{ik}(\omega)|\big]$ for $d_i > 0$.

(v) (MN_5) implies $\mu(A(\omega))$ = the largest eigenvalue of $\frac{1}{2}Q^{-1}[A^T(\omega)Q + QA(\omega)]$.

Remark 1.3.4. From Example 1.3.1, one can easily see that the logarithmic norm of a random matrix need not have to be positive real valued random variable. It can be any real valued random variable.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Remark 1.3.5. We note that the above discussion is also valid for random matrix functions $A(t,\omega)$. Furthermore, the logarithmic norm of a random matrix function preserves the regularity properties of $A(t,\omega)$. For example, if $A(t,\omega)$ is jointly measurable random matrix function, then the logarithmic norm $\mu(A(t,\omega))$ is also jointly measurable real-valued random function; if $A(t,\omega)$ is sample continuous random function, then the logarithmic norm $\mu(A(t,\omega))$ is sample continuous random function.

Remark 1.3.6. We also note that the $d_i > 0$ in (MN_4) and the positive definite matrix Q can be replaced by random numbers $d_i(\omega) > 0$ w.p. 1 and the positive definite random matrix $Q(\omega)$ w.p. 1.

The remainder of this section is devoted to apply the above presented discussion with regard to random matrices to dynamic processes in biological, chemical, engineering, and social systems described/approximated by the following systems of first order linear differential equations with random parameters:

$$x' = A(\omega)x, \qquad x_0(t_0,\omega) = x_0(\omega), \tag{1.3.11}$$

where $x \in R^n$, and $A(\omega)$ is $n \times n$ random matrix defined on a complete probability space $(\Omega, \mathcal{F}, P)$.

The goals of this part of the section are to find: (i) solution representation of (1.3.11) and (ii) its probability distributions in the context of the probability distributions of eigenvalues of $A(\omega)$. To fulfill these goals, we imitate the method of solving systems of first order deterministic differential equations by seeking the sample solution process of (1.3.11) in following form:

$$x(t,\omega) = c\exp[\lambda t], \tag{1.3.12}$$

where c is an unknown n-dimensional vector and λ is also an unknown variable. The goal of finding c and λ is based on the elementary procedure described in any elementary deterministic differential equations text-book. In this case, we obtain a n-th degree characteristic polynomial equation (1.3.3) associated with $A(\omega)$ in (1.3.11) with leading coefficient 1. The n values of λ's can be determined by solving this random polynomial equation. Then one can determine n unknown n-dimensional vectors c's corresponding to eigenvalues λ's by solving the systems of linear homogeneous algebraic equations (1.3.1).

When $A(\omega)$ possesses n distinct real eigenvalues $\lambda_1(A(\omega)), \lambda_2(A(\omega)), \ldots \lambda_i(A(\omega)), \ldots, \lambda_n(A(\omega))$ and its corresponding eigenvectors $c_1(\omega), c_2(\omega), \ldots, c_i(\omega), \ldots, c_n(\omega)$, then the sample fundamental matrix solution process of (1.3.11) is described by

$$\Phi(t, t_0, \omega) = [x_1(t,\omega), x_2(t,\omega), \ldots, x_i(t,\omega), \ldots, x_n(t,\omega)], \quad (1.3.13)$$

where $x_i(t,\omega)$'s are linearly independent solution processes of (1.3.11) represented by

$$x_i(t,\omega) = c_i(\omega)\exp[\lambda_i(A(\omega))(t-t_0)] \qquad \text{for } i \in I(1,n) = \{1, 2, ..., n\}.$$

Thus the sample solution process of (1.3.11) is represented by

$$x(t,\, \omega) = \Phi(t, t_0, \omega)x_0(\omega), \qquad \text{for } t \geq t_0. \qquad (1.3.14)$$

For a similar discussion with regard to repeated eigenvalues and complex eigenvalues of $A(\omega)$ can be carried out by imitating the deterministic work described in the text-book on elementary differential equations. This completes the discussion with regard to the solution representation goal of (1.3.11).

Now we focus our attention about the probability distribution goal of the solution process of (1.3.11) by finding the probability distribution functions of the eigenvalues of $A(\omega)$. For this purpose, let us rewrite the characteristic polynomial equation (1.3.3) associated with $A(\omega)$ in (1.3.11) as

$$p(z,\omega) = z^n + a_{n-1}(\omega)z^{n-1} + \cdots + a_0(\omega), \tag{1.3.15}$$

where for $i \in I(0, n-1) = \{0, 1, 2, \ldots, n-1\}$ the coefficients $a_i(\omega)$ are random variables determined by the random matrix $A(\omega)$.

For $i \in I(1,n) = \{1, 2, \ldots, n\}$, let $\lambda_i(\omega) = \lambda_i(A(\omega))$ are n roots of (1.3.15) and let

$$\begin{aligned}
\lambda_1(\omega) &= \alpha_1(\omega) + i\beta_1(\omega), \\
\lambda_2(\omega) &= \alpha_1(\omega) - i\beta_1(\omega), \\
\ldots\ldots & \quad \ldots\ldots \\
\lambda_{2k-1}(\omega) &= \alpha_k(\omega) + i\beta_k(\omega), \\
\lambda_{2k}(\omega) &= \alpha_k(\omega) - i\beta_k(\omega), \\
\lambda_{2k+1}(\omega) &= \tau_1(\omega), \\
\ldots\ldots & \quad \ldots\ldots \\
\lambda_n(\omega) &= \tau_{n-2k}(\omega),
\end{aligned}$$

where for $i \in I(1,k) = \{1, 2, \ldots, k\}$ and $j \in I(1, n-2k) = \{1, 2, \ldots, n-2k\}$, $\alpha_i(\omega)$, $\beta_i(\omega)$, and $\tau_j(\omega)$ are real valued random variables. For any real numbers u_i, v_i, $i \in I(1,n) = \{1, 2, \ldots, n\}$, the probability distributions of $\lambda_i(A(\omega))$ are determined by the following theorem.

Theorem 1.3.1. *If the random coefficients $a_i(\omega)$, $i \in I(1,n) = \{1, 2, \ldots, n\}$ of the characteristic polynomial equation of $A(\omega)$ in (1.3.15)*

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

have the joint distribution density $p(x_1, x_2, \ldots, x_n)$*, then for any real numbers* u_i *and* v_i*,* $i \in I(1,n) = \{1, 2, \ldots, n\}$*,*

$$P\big(\{Re\ \lambda_1(\omega) < u_1, Im\ \lambda_1(\omega) < v_{1,\ldots} Re\ \lambda_n(\omega) < u_n, Im\ \lambda_n(\omega) < v_n\}\big)$$
$$= \sum_{s=0}^{[n/2]} 2^s \int_{E_s} p(\Delta_1, \Delta_2, \ldots \Delta_n)\varphi(z_1, z_2, \ldots, z_n)$$
$$\times |\prod_{i>j}(z_i - z_j)| \prod_{i=1}^{s} dx_i\, dy_i \prod_{i=2s+1}^{n} dz_i \qquad (1.3.16)$$

where for $p \in I(1,s) = \{1, 2, \ldots, s\}$*,* $z_{2p-1} = x_p + iy_p$*,* $z_{2p} = x_p - iy_p$ *and for* $l \in I(2s+1, n) = \{2s+1, 2, \ldots, n\}$*,* z_l *are variables, the domain of integration* E_s *is defined by*

$$E_s = \{(x^T, y^T, z^T)^T \in R^s \times R^s \times R^{n-2s-1} :$$
$$x_1 < u_1, y_1 < v_1, \ldots, x_s < v_{2s}, -y_s < v_{2s}, z_l < u_l, 0 < v_l$$
$$\text{for } l \in I(2s+1, n)\},$$

and

$$\Delta_k = (-1)^k \sum_{i_1 < i_2 < \cdots < i_k} z_{i_1} z_{i_2} \cdots z_{i_k}$$

are the symmetric functions of the variables z_i*,* $i \in I(1,n) = \{1, 2, \ldots, n\}$*;* $\varphi(z_1, z_2, \ldots, z_n) = 1$*, if the values* z_i*,* $i \in I(1,n) = \{1, 2, \ldots, n\}$ *are ordered as described above, and* $\varphi(z_1, z_2, \ldots, z_n) = 0$ *otherwise.*

Remark 1.3.7. If the characteristic polynomial equation in (1.3.15) is of the following form

$$p(z, \omega) = z^n - a_{n-1}(\omega) z^{n-1} + \cdots + (-1)^{i-1} a_{n-i+1}(\omega) z^{n-i+1}$$
$$+ \cdots + (-1)^n a_0(\omega), \qquad (1.3.17)$$

where for $k \in I(0, n-1) = \{0, 2, \ldots, n-1\}$, $a_k(\omega)$ are complex valued random variables, and $\lambda_i(\omega)$ are roots of (1.3.17). If we assume that the real and imaginary parts of $a_k(\omega)$ are i.i.d. with $\mathcal{N}(0, \sigma)$, then the density function of the coefficients are given by

$$\left(\frac{1}{(2\pi)\frac{1}{2}\sigma}\right)^n \exp\left[-\frac{1}{\sigma^2}\sum_{k=0}^{n-1} a_k \overline{a}_k\right]$$

where $\overline{a}_k$'s are conjugates of a_k. Then the joint probability density function of the real and imaginary parts of $\lambda_i(\omega)$ of (1.3.17) is given by

$$\left(\frac{1}{(2\pi)\frac{1}{2}\sigma}\right)^{2n} \exp\left[-\frac{1}{2\sigma^2}\left[\sum_{k=1}^{n} z_k \sum_{k=1}^{n} \overline{z}_k + \cdots + z_1\overline{z}_1 \cdots z_n\overline{z}_n\right]\right]$$
$$\sum_{i=1}^{n}\sum_{j=i+1}^{n} |z_i - z_j|^2.$$

For further results about symmetric, non-symmetric, complex and real random matrices, we refer to Girko [118].

Finally, we conclude this section by presenting a result which gives an estimate for the solution process of (1.3.11) in context of the logarithmic norm of its coefficient matrix.

Theorem 1.3.2. *Let $x(t, \omega)$ be the solution process of* (1.3.11). *Then*

$$\|x(t, \omega)\| \le \|x_0(\omega)\| \exp[\mu(A(\omega))(t - t_0)], \quad \textit{for } t \ge t_0 \tag{1.3.18}$$

where $\mu(A(\omega))$ is a logarithmic norm as defined in (1.3.7).

Proof. For $x \in R^n$ and an arbitrary small $h > 0$,

$$\begin{aligned}\|(I + hA(\omega))x\| - \|x\| &\le \|(I + hA(\omega))\|\|x\| - \|x\| \\ &\le [\|(I + hA(\omega))\| - 1]\|x\|.\end{aligned}$$

This together with the definition of $\mu(A(\omega))$, yields

$$\lim_{h\to 0^+} \frac{1}{h}\Big[\|(I + hA(\omega))x\| - \|x\|\Big] \le \mu(A(\omega))\|x\|. \tag{1.3.19}$$

Let $x(t,\omega)$ be any sample solution of (1.3.11). From (1.3.19), we have

$$D^+\|x(t,\omega)\| \le \mu(A(\omega))\|x(t,\omega)\|, \quad \text{w.p. } 1 \tag{1.3.20}$$

with

$$\|x_0(\omega)\| \le u_0. \tag{1.3.21}$$

By solving the above inequality, we obtain

$$\|x(t,\omega)\| \le \|x_0(\omega)\| \exp[\mu(A(\omega))(t - t_0)], \qquad \text{for } t \ge t_0 \text{ w.p. } 1.$$

Thus proving the theorem.

1.4. STABILITY OF RANDOM MATRICES

In this section, we outline a few well-known results about the stability of random matrices. The results are direct extentions of the deterministic results that are summarized by Siljak [101]. Before presenting the stability results for random matrices, we first extend the definitions of a few well-known deterministics matrices.

Definition 1.4.1. An $n \times n$ random matrix $A(\omega)$ is called stable (or Hurwitz) if Re $\lambda_i(A(\omega)) < 0$ for all $i \in I(1,n) = \{1, 2, \ldots, n\}$ w.p. 1.

Definition 1.4.2. A random matrix $A(\omega)$ is called a Hicks matrix if all odd-order principle minors of $A(\omega)$ are negative and even-order principle minors of $A(\omega)$ are positive w.p.1.

Definition 1.4.3. An $n \times n$ random matrix $A(\omega) = (a_{ij}(\omega))$ is Metzler matrix if $a_{ij}(\omega) \ge 0$, $i \ne j$, for all $i, j \in I(1,n)$ w.p. 1.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Definition 1.4.4. An $n \times n$ random matrix $A(\omega) = (a_{ij}(\omega))$ is said to be dominant diagonal if

$$|a_{ii}(\omega)| > \sum_{j \neq i}^{n} |a_{ij}(\omega)|, \qquad \text{for every } i \in I(1,n) \text{ w.p. 1.} \tag{1.4.1}$$

If, in addition, $a_{ii}(\omega) > 0 (a_{ii}(\omega) < 0)$ for all $i \in I(1,n)$ then $A(\omega)$ is dominant positive (negative) diagonal matrix w.p.1.

Definition 1.4.5. An $n \times n$ random matrix $A(\omega) = (a_{ij}(\omega))$ is said to be quasidominant diagonal if there exist positive random variables $d_j(\omega)$, $j \in I(1,n)$ such that either

$$d_i(\omega)|a_{ii}(\omega)| > \sum_{j \neq i}^{n} d_j(\omega)|a_{ij}(\omega)|, \qquad \text{for every } i \in I(1,n) \text{ w.p. 1.} \tag{1.4.2}$$

or

$$d_j(\omega)|a_{jj}(\omega)| > \sum_{i \neq j}^{n} d_i(\omega)|a_{ij}(\omega)|, \qquad \text{for every } j \in I(1,n) \text{ w.p.1.} \tag{1.4.3}$$

Definition 1.4.6. An $n \times n$ random matrix $A(\omega)$ is called positive definite if $x^T A(\omega) x > 0$ for all n-dimensional vectors $x \neq 0$.

We are ready to formulate the results concerning the stability of random matrices.

Theorem 1.4.1. *A random Metzler $A(\omega)$ is stable iff it is Hicks.*

Theorem 1.4.2. *A random Metzler $A(\omega) = (a_{ij}(\omega))$ is stable iff*

$$(-1)^k \begin{vmatrix} a_{11}(\omega) & a_{12}(\omega) & \cdots & a_{1k}(\omega) \\ a_{21}(\omega) & a_{22}(\omega) & \cdots & a_{2k}(\omega) \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1}(\omega) & a_{k2}(\omega) & \cdots & a_{kk}(\omega) \end{vmatrix} > 0,$$

$$\textit{for every } k \in I(1,n) \textit{ w.p.1.} \tag{1.4.4}$$

Theorem 1.4.3. *A random Metzler $A(\omega)$ is stable iff it is quasidominant negative diagonal.*

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Theorem 1.4.4. *A random $A(\omega)$ is stable iff for any positive definite random symmetric matrix $G(\omega)$ there is a positive definite random symmetric matrix $H(\omega)$ such that*

$$A^T(\omega)H(\omega) + H(\omega)A(\omega) = -G(\omega). \tag{1.4.5}$$

The Definition 1.4.1 of a stability of a random matrix $A(\omega)$ is too restrictive. However, if we know the probability density function of a random matrix, one can modify Definition 1.4.1. The modification is as follows:

Definition 1.4.7. An $n \times n$ random matrix $A(\omega)$ is said to be stable with the following probability

$$P\Big(\{\omega : \text{Re } \lambda_i(A(\omega)) < 0 \text{ for all } i \in I(1,n)\}\Big). \tag{1.4.6}$$

In the following, we simply present the results concerning the computation of the probability in (1.4.6). More details, we refer to Girko [118] and Bharucha-Reid and Sambandham [8].

Theorem 1.4.5. *Let $A(\omega)$ be a random symmetric matrix of order n with probability density $p(x)$ and let $\lambda_i(A(\omega))$'s be its eigenvalues. Then,*

$$\begin{aligned} P\big(\{\omega : \mathit{Re}\ \lambda_i(A(\omega)) < 0 \text{ for all } i \in I(1,n)\}\big) \\ = c\int p\big(-Z_{n\times(n+1)} Z^T_{n\times(n+1)}\big)dZ_{n\times(n+1)} \end{aligned} \tag{1.4.7}$$

where $Z_{n\times(n+1)}$ is $n \times (n+1)$ real matrix, and

$$c = \pi^{-n(n+3)/4} \prod_{i=1}^{n} \Gamma[(i+1)/2], \qquad dZ_{n\times(n+1)} = \prod_{i=1}^{n} \prod_{j=1}^{(n+1)} dZ_{ij}.$$

Theorem 1.4.6. *Let $A(\omega)$ be a random nonsymmetric matrix with a joint probability density function $p(x)$ and $\lambda_i(A(\omega))$'s be its eigenvalues. Then,*

$$P\big(\{\omega : Re\ \lambda_i(A(\omega)) < 0 \text{ for all } i \in I(1,n)\}\big)$$
$$= 2^n \int_0^{+\infty} \cdots \int_0^{+\infty} \prod_{i=1}^{n} dy_{ii}$$
$$\int \cdots \int \prod_{i=1}^{n} y_{ii}^i |I(YY^*, H)| p\big(YY^*(H - I2^{-1})\big) \prod_{i<j} dy_{ij} dh_{ij}, \tag{1.4.8}$$

where $H = -H^* = (h_{ij})$ is a $n \times n$ skew-symmetric matrix, Y is a $n \times n$ triangular matrix $y_{ii} > 0$, $i \in I(1,n)$, $J(X,H)$ is the Jacobian of the matrix transform $A = X(H - 2^{-1}I)$, and X is a $n \times n$ symmetric matrix.

Remark 1.4.1. The random companion matrix associated with the random characteristic polynomial equation (1.3.15) is defined by

$$C(\omega) = \begin{bmatrix} -a_{n-1}(\omega) & -a_{n-1}(\omega) & \cdots & -a_1(\omega) & -a_0(\omega) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{1.4.9}$$

It has been shown that the average of the zeros of (1.3.15) and the average of the eigenvalues of $C(\omega)$ are very close to each other [8].

For more related results concerning the stability properties of random polynomials we refer to Girko [118] and Bharucha-Reid and Sambandham [8].

1.5. APPLICATIONS

In this section we present some applications of random polynomials and the corresponding error estimates.

a) Economic Analysis of Capital and Investment

In the first application we consider an example from economics. In the economic analysis of capital and investment the present or discounted value, v, of a stream of anticipated costs and returns associated with an investment is given by the present value formula

$$P(c) = a_0 + a_1\left(\frac{1}{1+c}\right) + \cdots + a_n\left(\frac{1}{1+c}\right)^n, \tag{1.5.1}$$

where a_i is the net return in period i, $i = 0, 1, \ldots, n$ which may be positive or negative and c is the rate of discount. Typically c is a positive number between 0 and 1/2. The effect of the transformation (1.5.1) is to weight a future return a_i by the fraction $1/(1+c)^i$. One economic justification for so doing is that $a_i/(1+c)^i$ available at time 0 might be reinvested at a rate c each period to yield $\frac{a_i}{(1+c)^i}(1+c)^i = a_i$ at the end of i periods. In this sense a_i at the end of i periods and $a_i/(1+c)^i$ at time 0 are equivalent. If we denote $1/(1+c)$ by x then (1.5.1)can be rewritten as

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, \tag{1.5.2}$$

where x and c are related inversely with values of c between 0 and 1 corresponding to x between 1/2 and 1. If the returns a_i are uncertain (1.5.2) becomes a random polynomial.

The typical type profile of returns for a capital or investment project is that early returns are investment costs (negative a_i) while the later returns are earnings (positive a_i). For $c = 0$ or $x = 1$ the present value is just the sum of all the returns, which will be positive if the project is economically viable. As the discount rate c is increased the later positive returns are reduced by higher discount factors, $1/(1+c)^i$, than are early negative returns, so that the present

value $p(x)$ is a decreasing function of c. A value of c for which $p(x)$ is "discounted to zero" is called a rate of return of the investment. Such a value of c is a root of $p(c)$ and $x = 1/(1+c)$ is a root of $p(x)$.

In most cases there will be a unique real root of (1.5.2) for x between 1/2 and 1, corresponding to rates of return c between 0 and 1 or 0% and 100%. In this case c is called "the" rate of return of the investment where the root is not unique an economic analysis might turn on other considerations. For a discussion of issues we refer to Masse [92].

When (1.5.2) satisfies the hypothesis of Theorem 1.2.1, we get

$$|E[\xi(a)] - \xi(E[a])| < ER(\Delta),$$

where $ER(\Delta)$ is derived in (1.2.7).

b) Free Damped Motion of Spring

As a second application we consider the effect of the resistance of the medium upon the mass on the spring. We assume that no external forces are present, that is, we consider the case of free damped motion. Then the basic differential equation of free damped motion reduces to

$$m\,\frac{d^2x}{dt^2} + a\,\frac{dx}{dt} + kx = 0 \tag{1.5.3}$$

and therefore

$$\frac{d^2x}{dt^2} + b_1\,\frac{dx}{dt} + b_2x = 0. \tag{1.5.4}$$

Let b_1 and b_2 be random variables and $b_1^2 - 4b_2 > 0$ with probability one. Then the auxiliary equation of (1.5.4) is given by

$$\gamma^2 + b_1\gamma + b_2 = 0 \tag{1.5.5}$$

which has two real zeros. Therefore the general solution of (1.5.5) is

$$x = c_1e^{\gamma_1 t} + c_2e^{\gamma_2 t}, \tag{1.5.6}$$

where γ_1, γ_2 are two real roots of (1.5.5). When (1.5.6) satisfies hypothesis of Theorem 1.2.2 we get

$$|E[\gamma_i(b)] - \gamma_i(E[b])| < E[R(\Delta)], \qquad i = 1, 2 \tag{1.5.7}$$

where $E[R(\Delta)]$ is derived in Theorem 1.2.2; and $\gamma_i(E[b])$ is the solution of

$$r^2 + E[b_1]r + E[b_2] = 0. \tag{1.5.8}$$

The differential equation corresponding to (1.5.8) can be described as

$$\frac{d^2m}{dt^2} + E[b_1]\frac{dm}{dt} + E[b_2]m = 0. \tag{1.5.9}$$

This is a deterministic differential equation corresponding to (1.5.4). The relation (1.5.8) can provide some information about the solutions of (1.5.4) and (1.5.9).

1.6. NUMERICAL EXAMPLES

In this section we illustrate the scope and usefulness of the results in Section 1.1–1.2 by exhibiting a few numerical examples. Let

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0, \tag{1.6.1}$$

$$x^n - 1 = 0, \tag{1.6.2}$$

where (1.6.1) is a random algebraic polynomial, with

$$a_0 \in N(-1, \sigma), \quad a_n \in N(1, \sigma), \quad \text{and}$$
$$a_i \in N(0, \sigma), \quad i = 1, \dots, n-1 \tag{1.6.3}$$

where $N(m, \sigma)$ represents normal distribution with mean m and standard deviation σ; (1.6.2) is an algebraic polynomial obtained by replacing a_i in (1.6.1) by the corresponding mean of a_i.

Example 1.6.1. Consider (1.6.1) and (1.6.2) with $n = 19$. That is

$$a_0 + a_1 x + \cdots + a_{19} x^{19} = 0 \tag{1.6.4}$$

and

$$x^{19} - 1 = 0, \tag{1.6.5}$$

where a_i are independent normally distributed as in (1.6.3). Then from (1.2.7) we obtain

$$|E[\xi(a)] - \xi(E[a])| \leq E[R(\Delta)], \tag{1.6.6}$$

where

$$E[R(\Delta)] = 2 \max_{0 \leq i \leq 19} E\left[\chi'_\Delta |a_i|\right] + E\left[\chi_\Delta\right] |1 - \xi(E[a])| + L(I_2^{20} - I_3^{20}),$$

$$L = 1 + \max_{0 \leq i \leq 19} |R + E[a_i]|,$$

$$I_2 = E\left[\chi_\Delta / (1 - t^2)\right],$$

$$I_3 = E\left[\chi_\Delta\right],$$

$$t = |z_1 - E[a_1]| R,$$

R is the radius of a suitably large poly disc centered at $E[a_i]$, that is

$$H_R = \{z \in C^n : |x - E[a_i]| < R, i = 0, 1, \ldots 19\}$$

$0 < \Delta < R$. For a sample of size 10, we obtained the roots of the sample polynomial (1.6.4) and the average polynomial (1.6.5). These results are presented in Fig. 1.6.1 and Fig. 1.6.2. In Fig. 1.6.1 x's represent the roots of the average polynomial and t's represent the roots of the random polynomial.

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

In Fig. 1.6.2 x's represent the roots of the average polynomial and t's represent the average of the roots of random polynomials. Table 1.6.1 contains the roots of average polynomial and mean and standard deviation of the random polynomial. Table 1.6.2 contains the deflection of the roots of average polynomial to the average of the roots of random polynomial. Table 1.6.3 contains the theoretical upper bound (1.6.6) of the difference between the average of the solution and solution of the average equation.

Example 1.6.2. Consider (1.6.4), (1.6.5) where a_i are dependent random variables and normal distributed as in (1.6.3) with correlation between any two random variables is 0.5. Figs. 1.6.3 and 1.6.4 are analogs of Figs. 1.6.1 and 1.6.2 respectively for dependent normal random variables and Tables 1.6.4, 1.6.5 and 1.6.6 are analogs of Tables 1.6.1, 1.6.2 and 1.6.3 respectively for dependent normal random variables.

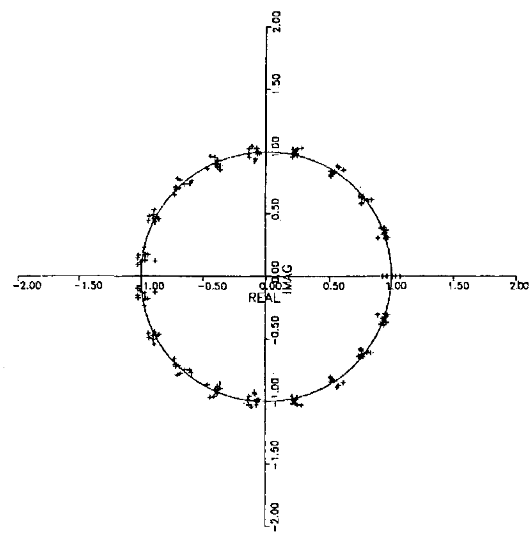


FIGURE 1.6.1

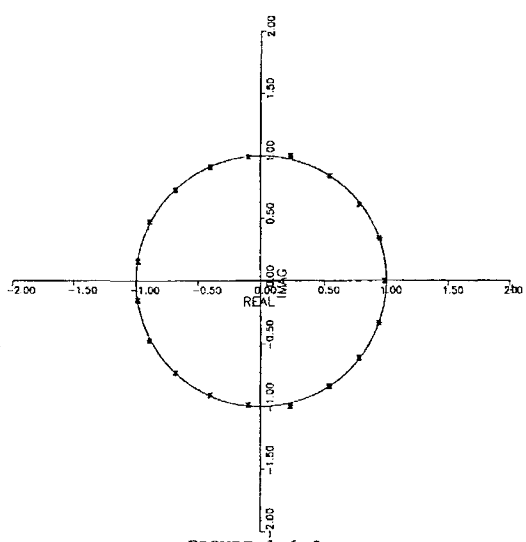


FIGURE 1.6.2

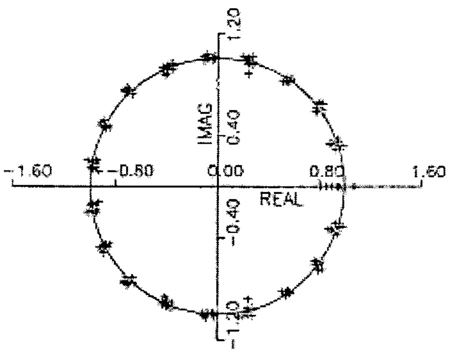


FIGURE 1.6.3

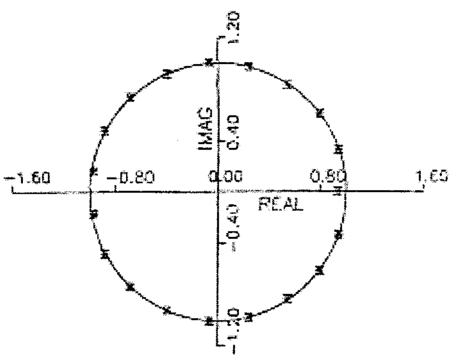


FIGURE 1.6.4

Table 1.6.1 The Zeros of Average Polynomial

Real Part	Imaginary Part	Real Part	Imaginary Part
0.9909	0.0000	−0.9881	−0.1642
0.9416	0.3363	−0.8914	−0.4746
0.7827	0.6157	−0.6780	−0.7352
0.5435	0.8365	−0.4018	−0.9094
0.2363	0.9966	−0.0971	−0.9921
−0.0971	0.9921	0.2363	−0.9966
−0.4018	0.9094	0.5435	−0.8365
−0.6780	0.7352	0.7827	−0.6157
−0.8914	0.4746	0.9416	−0.3363
−0.9881	0.1642		

Table 1.6.2 Average Roots

Real Part	Imaginary Part	Standard Deviation	Euclidean Distance
0.9866	0.0000	0.0446	0.0043
0.9433	0.3348	0.0423	0.0022
0.7804	0.6137	0.0371	0.0030
0.5445	0.8384	0.0414	0.0021
0.2344	0.9970	0.0329	0.0019
−0.0953	0.9888	0.0506	0.0037
−0.3999	0.9098	0.0494	0.0019
−0.6807	0.7302	0.0629	0.0056
−0.8919	0.4742	0.0393	0.0006
−0.9849	0.1537	0.0609	0.0109
−0.9849	−0.1537	0.0609	0.0109
−0.8919	−0.4742	0.0393	0.0006
−0.6807	−0.7302	0.0629	0.0056
−0.3999	−0.9098	0.0494	0.0019
−0.0953	−0.9888	0.0506	0.0037
0.2344	−0.9970	0.0329	0.0019
0.5445	−0.8384	0.0414	0.0021
0.7804	−0.6137	0.0371	0.0030
0.9433	−0.3348	0.0423	0.0022

Table 1.6.3 Estimated Deflection

Δ	$ER(\Delta)$	$ER_2(\Delta)$	$ER_4(\Delta)$
0.22	0.1725	0.1699	0.1697
0.23	0.1656	0.1614	0.1609
0.24	0.1606	0.1538	0.1530
0.25	0.1581	0.1477	0.1463
0.26	0.1590	0.1440	0.1445
0.27	0.1660	0.1436	0.1395
0.28	0.1876	0.1479	0.1413
0.29	0.2163	0.1585	0.1484

Table 1.6.4 The Roots of the Average Polynomial

Real Part	Imaginary Part
.9479776	0.0000000
.9385939	.3304983
.7929141	.6154910
.5444317	.8274597
.2418973	.9768263
−.0678546	1.0020031
−.3957972	.9160294
−.6833067	.7380051
−.8846311	.4762088
−.9729657	.1703126
−.9729657	−.7103126
−.8846311	−.4762088
−.6833067	−.7380051
−.3957972	−.9160294
−0.678546	−1.0020031
.2418973	−.9768263
.5444317	−.8274597
.7929141	−.6154910
.9385939	−.3304983

Table 1.6.5 The Average of the Roots

Average Roots Real Part	Imaginary Part	Standard Deviation
.9353027	0.0000000	.0768116
.9416229	.3299450	.0397804
.7950052	.6133575	.0354159
.5434202	.8275003	.0214688
.2416081	.9723315	.0453742
−.0675997	1.0043470	.0370409
−.3948250	.9150530	.0356639
−.6857177	.7372559	.0347107
−.8823378	.4740675	.0304207
−.9715570	.1651164	.0394782
−.9715570	−.1651164	.0394782
−.8823378	−.4740675	.0304207
−.6857177	−.7372559	.0347107
−.3948250	−.9150530	.0356639
−.0675997	−1.0043470	.0370409
.2416081	−.9723315	.0453742
.5434202	−.8275003	.0214688
.7950052	−.6133575	.0354159
.9416229	−.3299450	.0397804

Table 1.6.6 The Value of $ER(\Delta)$ for a Given Δ

Δ	$E(R(\Delta))$
0.22	0.2979
0.23	0.3005
0.24	0.3036
0.24	0.3073
0.25	0.3115
0.26	0.3163
0.27	0.3218
0.28	0.3277
0.28	0.3343

1.7. NOTES AND COMMENTS

The calculation of the difference between the expectation of the random zeros of a random algebraic polynomial and the zero of the corresponding averaged polynomial is called an average problem (Bharucha-Reid and Sambandham [8]). An upper bound on the absolute deviation between the value of a random multivariate function and the value of the deterministic function is established in Section 1.1. Lemma 1.1.1 is an extension of a result of Christensen and Bharucha-Reid [26]. For further discussion, see Bharucha-Reid and Sambandham [8]. The content of Section 1.2 is adapted from Christensen and Bharucha-Reid [26] and Ladde and Sambandham [71]. See also Bharucha-Reid and Sambandham [8]. The material of Section 1.3 is based on the work of Girko [118] and Ladde [60]. The content of Section 1.4 is based on the material in Siljak [101], Girko [118] and Bharucha-Reid and Sambandham [8]. For further details, see Girko [118], Ladde [60], Bharucha-Reid and Sambandham [8]. The material on applications, Section 1.5, is based on Masse [92] and elementary applications in differential equations [119]. For further details, see Bharucha-Reid and Sambandham [8]. The numerical examples in Section 1.6 are taken from Ladde and Sambandham [71]. For further information about random polynomials, see Farahmond [32], Hamblen [44], Von Scheidt and Bharucha-Reid [109], Bharucha-Reid and Sambandham [8] and references cited in above mentioned references.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

CHAPTER 2: ORDINARY DIFFERENTIAL SYSTEMS WITH RANDOM PARAMETERS

2.0 INTRODUCTION

The concept of system of differential equations with random parameters generates a very general and difficult problem of "stochastic versus deterministic" (or "randomness versus non-randomness"). Roughly speaking, this means that to what extent the solution processes of systems of differential equations with random parameters deviate from the solution processes of corresponding systems of differential equations with deterministic parameters. In this chapter, three major techniques for studying nonlinear initial value problems are developed and a solution to the above problem is addressed.

A method of a generalized variation of constants for a nonlinear system of differential equations with random parameters is discussed in Section 2.1. By employing the concept of Lyapunov-like functions and the theory of system of random differential inequalities, several variational comparison theorems along with several examples are presented in Section 2.2. In Section 2.3, by applying Liouville's theorem in the theory of dynamic systems, the Liouville-type theorem for finding the joint probability density function of the solution process of the system of differential equations with random parameters is outlined. Section 2.4 deals with the pth moment stability results via the methods of comparison and variation of constants. Moreover, the results provide estimates on the solution processes. By employing the methods of comparison and variation of constants, estimates for the absolute pth mean deviation of a solution process of a system of

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

differential equations with random parameters with a solution process of the corresponding deterministic system of differential equations are developed in Section 2.5. The pth mean stability results of solution process of a system of differential equations with random parameters relative to a solution process of a corresponding deterministic system of differential equations are proved in Section 2.6. A mathematical n-species community model under random environmental fluctuations illustrating the scope and usefulness of the mathematical results is discussed in Section 2.7. The scope and the significance of mathematical results are further illustrated by numerical examples in Section 2.8.

2.1. VARIATION OF CONSTANTS METHOD

The method of variation of constants is an important technique in studying the qualitative properties of the solutions of a nonlinear system of stochastic differential equations. In this section, we present a generalized variation of constants formula for stochastic systems of differential equations with random parameters.

We consider the stochastic initial value problem (SIVP) with random parameters

$$y' = F(t, y, \omega), \qquad y(t_0, \omega) = y_0(\omega) \tag{2.1.1}$$

and corresponding deterministic initial value problem (DIVP)

$$m' = \widehat{F}(t, m), \qquad m(t_0) = m_0 = E[y_0(\omega)], \tag{2.1.2}$$

which is obtained by ignoring the random disturbances in the system described by (2.1.1). In our presentation we utilize the following random initial value problem (RIVP)

$$x' = \widehat{F}(t, x), \qquad x(t_0, \omega) = x_0(\omega). \tag{2.1.3}$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

In (2.1.1), (2.1.2) and (2.1.3), y, m, x and $\widehat{F} \in R^n$; $m_0 = E[y_0(\omega)]$, and E stands for expectation of a random variable; $F \in M[R_+ \times R^n, R[\Omega, R^n]]$ and $F(t, y, \omega)$ sample continuous in y for fixed $t \in R_+$; $\widehat{F} \in C[R_+ \times R^n, R^n]$. We assume that

(H_1) F satisfies desired regularity conditions so that the initial value problem (2.1.1) has a sample solution process existing for $t \geq t_0$;

(H_2) the Jacobian of $\widehat{F}$, $\widehat{F}_x$ exists, and $\widehat{F}_x \in C[R_+ \times R^n, R^{n^2}]$.

The assumption (H_2) implies that $\bar{x}(t) = x(t, t_0, z_0)$ is a unique solution process of either (2.1.2) or (2.1.3) depending on the choice of z_0. Moreover, $x(t, t_0, z_0)$ is sample continuously differentiable with respect to (t_0, z_0) and $\frac{\partial x}{\partial x_0}(t, t_0, z_0) = \Phi(t, t_0, z_0)$ is the fundamental matrix solution process of the variational system

$$z' = \widehat{F}_x(t, \bar{x}(t))z, \qquad \Phi(t_0, t_0, z_0) = I_{n \times n} \tag{2.1.4}$$

associated with (2.1.3). For further details about the fundamental properties of sample solution processes of (2.1.1), we refer to Ladde and Lakshmikantham [67]. A remark similar to the above statement can be made relative to the solution $m(t) = m(t, t_0, m_0)$ of (2.1.2).

Hereafter, without further mention, it is assumed that all inequalities and relations involving random quantities are valid with probability one (w.p. 1).

We shall now formulate a few basic results that relate a solution process of (2.1.1) with solution processes of (2.1.2) or (2.1.3).

Theorem 2.1.1. *Let the hypotheses* (H_1)–(H_2) *be satisfied, and let* $y(t, \omega) = y(t, t_0, y_0(\omega), \omega)$ *and* $x(t, \omega) = x(t, t_0, x_0(\omega))$ *be the sample solution processes of* (2.1.1) *and* (2.1.3) *respectively, existing for* $t \geq t_0$ *with* $x_0(\omega) = y_0(\omega)$. *Further assume that* $V \in C[R_+ \times R^n, R[\Omega, R^m]]$

and its sample partial derivative $V_x(t, y, \omega)$ exists and is continuous for $(t, x) \in R_+ \times R^n$. Then

$$V(t, y(t,\omega), \omega) = V(t_0, x(t,\omega), \omega) + \int_{t_0}^{t} [V_s(s, x(t, s, y(s,\omega)), \omega)$$
$$+V_x(s, x(t, s, y(s,\omega)), \omega)\Phi(t, s, y(s,\omega))R(s, y(s,\omega), \omega)]ds, \quad (2.1.5)$$

where $R(t, y, \omega) = F(t, y, \omega) - \widehat{F}(t, y)$.

Proof. From (2.1.1) and (2.1.3), system (2.1.1) can be rewritten as

$$y' = \widehat{F}(t, y) + R(t, y, \omega), \qquad y(t_0, \omega) = y_0(\omega), \quad (2.1.6)$$

where $R(t, y, \omega)$ is as defined in (2.1.5). Let $x(t, s, y(s,\omega))$ and $x(t,\omega) = x(t, t_0, y_0(\omega))$ be the sample solution processes of (2.1.3) through $(s, y(s,\omega))$ and $(t_0, y_0(\omega))$, respectively. Let $y(s,\omega) = y(s, t_0, y_0(\omega), \omega)$ be the sample solution of (2.1.1) through $(t_0, y_0(\omega))$. Now we compute the total sample derivative of $V(s, x(t, s, y(s,\omega)), \omega)$ with respect to s as

$$\begin{aligned}
\frac{dV}{ds}&(s, x(t, s, y(s,\omega)), \omega) = V_s(s, x(t, s, y(s,\omega)), \omega) \\
&\quad + V_x(s, x(t, s, y(s,\omega)), \omega)\left[\frac{d}{ds}x(t, s, y(s,\omega))\right] \\
&= V_s(s, x(t, s, y(s,\omega)), \omega) + V_x(s, x(t, s, y(s,\omega)), \omega) \\
&\quad \times \left[-\Phi(t, s, y(s,\omega))\widehat{F}(s, y(s,\omega)) + \Phi(t, s, y(s,\omega))y'(s,\omega)\right] \\
&= V_s(s, x(t, s, y(s,\omega)), \omega) \\
&\quad + V_x(s, x(t, s, y(s,\omega)), \omega)\Phi(t, s, y(s,\omega))R(s, y(s,\omega), \omega) \text{ w.p. } 1.
\end{aligned}$$

Here we have used Theorem A.2.1 to simplify the expression. Integrating (in the sample sense) both sides with respect to s, from t_0 to t, and noting $x(t, t, y(t, t_0, y_0(\omega), \omega)) = y(t, t_0, y_0(\omega), \omega)$, we obtain

$$V(t, y(t,\omega), \omega) = V(t_0, x(t,\omega), \omega) + \int_{t_0}^{t} [V_s(s, x(t, s, y(s,\omega)), \omega)$$
$$+ V_x(s, x(t, s, y(s,\omega)), \omega)\Phi(t, s, y(s,\omega))R(s, y(s,\omega), \omega)]ds.$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

This completes the proof of the theorem.

In the following, we shall demonstrate the scope and significance of the preceding theorem.

Corollary 2.1.1. *Let the assumptions of* Theorem 2.1.1 *be satisfied except* $V(t,x,\omega)$ *is replaced by* $V(t,x,\omega) = x$. *Then*

$$y(t,\omega) = x(t,\omega) + \int_{t_0}^{t} \Phi(t,s,y(s,\omega))R(s,y(s,\omega),\omega)ds. \tag{2.1.7}$$

This corollary is a well-known (Ladde and Lakshmikantham [67]) nonlinear variation of constants formula for systems of stochastic differential equations.

Problem 2.1.1. If $V(t,x,\omega) = \|x\|^2$ then (2.1.5) in Theorem 2.1.1 reduces to

$$\begin{aligned}\|y(t,\omega)\|^2 = \|x(t,\omega)\|^2 + 2\int_{t_0}^{t} & x^T(t,s,y(s,\omega)) \\ & \times\Phi(t,s,y(s,\omega))R(s,y(s,\omega),\omega)ds. \end{aligned}\tag{2.1.8}$$

The following result provides an expression for the difference between the solution processes of (2.1.1) with solution processes of (2.1.2) or (2.1.3).

Theorem 2.1.2. *Suppose all the hypotheses of* Theorem 2.1.1 *hold. Then*

$$\begin{aligned} V(t,y(t,\omega) - \bar{x}(t),\omega) &= V(t_0,x(t,\omega) - \bar{x}(t),\omega) \\ &+ \int_{t_0}^{t}\Big[V_s(s,x(t,s,y(s,\omega)) - x(t,s,\bar{x}(s)),\omega) + V_x(s,x(t,s,y(s,\omega)) \\ &\qquad -x(t,s,\bar{x}(s)),\omega)\Phi(t,s,y(s,\omega))R(s,y(s,\omega),\omega)\Big]ds \end{aligned}\tag{2.1.9}$$

where $\bar{x}(t) = x(t,t_0,z_0)$ *is the solution process of either* (2.1.2) *or* (2.1.3) *depending on the choice of* z_0.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Proof. By following the proof of Theorem 2.1.1, we have the relation

$$\frac{dV}{ds}(s, x(t, s, y(s, \omega)) - \bar{x}(t, s, \bar{x}(s)), \omega)$$
$$= V_s(s, x(t, s, y(s, \omega)) - \bar{x}(t, s, \bar{x}(s)), \omega)$$
$$+ V_x(s, x(t, s, y(s, \omega)) - \bar{x}(t, s, \bar{x}(s)), \omega)\Phi(t, s, y(s, \omega))R(s, y(s, \omega), \omega).$$

By integrating the above relation with respect to s, from t_0 to t and noting the fact that $x(t, t, y(t, \omega)) = y(t, \omega)$, we complete the proof of the theorem.

Problem 2.1.2. If $V(t, x, \omega) = \|x\|^2$ then (2.1.9) in Theorem 2.1.2 reduces to

$$\|y(t, t_0, y_0(\omega), \omega) - x(t, t_0, z_0)\|^2 = \\ \|x(t, t_0, y_0(\omega)) - x(t, t_0, z_0)\|^2 + 2\int_{t_0}^{t}(x(t, s, y(s, \omega)) \\ -x(t, s, \bar{x}(s)))^T \Phi(t, s, y(s, \omega))R(s, y(s, \omega), \omega)ds. \quad (2.1.10)$$

We recall that $\bar{x}(t) = x(t, t_0, z_0)$ is the solution process of either (2.1.2) or (2.1.3) depending on the choice of z_0. In other words $\bar{x}(t)$ is either $m(t) = m(t, t_0, m_0) = x(t, t_0, m_0)$ or $x(t, t_0, x_0(\omega))$.

Remark 2.1.1. Another variation of Theorem 2.1.2 can be formulated, analogously. Its proof follows by following the argument used in the proofs of Theorems 2.1.1 and 2.1.2. This result is formulated as problem.

Problem 2.1.3. Assume that all the hypotheses of Theorem 2.1.2 are satisfied. Then

$$V(t, y(t, \omega) - \bar{x}(t), \omega) = \\ V(t_0, x(t, t_0, y_0(\omega) - z_0), \omega) + \int_{t_0}^{t}\Big[V_s\Big(s, x(t, s, y(s, \omega) - \bar{x}(s)), \omega\Big) \\ +V_x(s, x(t, s, y(s, \omega) - \bar{x}(s)), \omega)\Phi(t, s, y(s, \omega) - \bar{x}(s)) \\ \widetilde{R}(s, y(s, \omega), \bar{x}(s), \omega)ds$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

where

$$\widetilde{R}(s, y, x, \omega) = F(s, y, \omega) - \widehat{F}(s, x) - \widehat{F}(s, y - x);$$

$x(t, t_0, y_0(\omega) - z_0)$ is the solution process of (2.1.3) through $(t_0, y_0(\omega) - z_0)$ and z_0 is as defined in Theorem 2.1.2.

To further illustrate the scope and usefulness of Theorems 2.1.1 and 2.1.2, let us assume certain regularity conditions which will translate (2.1.1), (2.1.2), and (2.1.3) into a suitable form for our discussions. We suppose that $F(t, 0, \omega) \equiv 0$, the sample derivative $\frac{\partial F}{\partial y}(t, y, \omega)$ of $F(t, y, \omega)$ exists, and it is sample continuous in y for each $t \in R_+$. From this and Lemma A.2.1, (2.1.1) can be rewritten as

$$y' = A(t, y, \omega)y, \qquad y(t_0, \omega) = y_0(\omega) \tag{2.1.11}$$

where

$$A(t, y, \omega) = \int_0^1 \left(\frac{\partial}{\partial y} F(t, sy, \omega) \right) ds. \tag{2.1.12}$$

Similarly, one can assume that $\widehat{F}(t, 0) \equiv 0$. This together with the continuous differentiability of $\widehat{F}(t, x)$ in x, one can rewrite (2.1.2) and (2.1.3) as

$$m' = \widehat{A}(t, m)m, \qquad m(t_0) = m_0, \tag{2.1.13}$$

and

$$x' = \widehat{A}(t, x)x, \qquad x(t_0, \omega) = x_0(\omega), \tag{2.1.14}$$

where

$$\widehat{A}(t, x) = \int_0^1 \left(\frac{\partial \widehat{F}}{\partial x}(t, sx) \right) ds. \tag{2.1.15}$$

From Lemma A.2.2, we note that

$$x(t, s, y(s, \omega)) - x(t, s, \bar{x}(s)) = \psi(t, s, y(s, \omega), \bar{x}(s))(y(s, \omega) - \bar{x}(s)), \tag{2.1.16}$$

where

$$\psi(t,s,y(s,\omega),\bar{x}(s)) = \int_0^1 \Phi(t,s,\bar{x}(s)+u(y(s,\omega)-\bar{x}(s)))du.$$

Remark 2.1.2. In the light of the above discussion, relation (2.1.9) in the context of $V(t,x,\omega)=\|x\|^2$ reduces to

$$\begin{aligned}\|y(t,t_0,y_0(\omega),\omega)-x(t,t_0,z_0)\|^2 &= \|x(t,t_0,y_0(\omega))-x(t,t_0,z_0)\|^2\\ &+2\int_{t_0}^t \Big(y(s,\omega)-\bar{x}(s,\omega)\Big)^T \psi^T(t,s,y(s,\omega),\bar{x}(s,\omega))\\ &\times \Phi(t,s,y(s,\omega))R(s,y(s,\omega),\omega)ds. \qquad (2.1.17)\end{aligned}$$

Example 2.1.1. Let us consider

$$y' = A(t,\omega)y, \qquad y(t_0,\omega)=y_0(\omega), \qquad (2.1.18)$$

$$m' = \widehat{A}(t)m, \qquad m(t_0)=m_0=E[y_0(\omega)], \qquad (2.1.19)$$

and

$$x' = \widehat{A}(t)x, \qquad x(t_0,\omega)=x_0(\omega), \qquad (2.1.20)$$

where $A(t,\omega)$ is an $n\times n$ random matrix function which is smooth enough to assure the existence of sample solution processes for $t\ge t_0$, and $\widehat{A}(t)$ is a rate matrix which is obtained by neglecting the randomness in the system. In particular, $\widehat{A}(t)=E[A(t,\omega)]$ if $E[A(t,\omega)]$ exists. In this case, $x(t,t_0,x_0(\omega))=\Phi(t,t_0)x_0(\omega)$, where $\Phi(t,t_0)$ is the fundamental matrix solution process of either (2.1.19) or (2.1.20). Note that in the linear case $\psi(t,s,y(s,\omega),\bar{x}(s))=\Phi(t,s)$. With regard to (2.1.18), (2.1.19) and (2.1.20) in the context of $V(t,x,\omega)=\|x\|^2$, (2.1.15), (2.1.16) and (2.1.4), (2.1.8) and (2.1.17) reduce to

$$\begin{aligned}\|y(t,\omega)\|^2 &= \|x(t,\omega)\|^2\\ &+2\int_{t_0}^t y^T(s,\omega)\Phi^T(t,s)\Phi(t,s)[A(s,\omega)-\widehat{A}(s)]y(s,\omega)ds, \qquad (2.1.21)\end{aligned}$$

$$\|y(t,\omega)-\bar{x}(t)\|^2 = \|x(t,\omega)-\bar{x}(t)\|^2$$
$$+\,2\int_{t_0}^{t}\Big(y(s,\omega)-\bar{x}(s)\Big)^T \Phi^T(t,s)\Phi(t,s)[A(s,\omega)-\widehat{A}(s)]y(s,\omega)ds, \tag{2.1.22}$$

respectively.

2.2. COMPARISON METHOD

Historically, the Lyapunov second method has played a very significant role in the qualitative and quantitative analysis of systems of differential equations. In the following, by employing the concept of random vector Lyapunov-like functions and the theory of random differential inequalities, we shall present comparison theorems which have wide range of applications in the theory of error estimates and stability analysis of systems of stochastic differential equations.

We assume that $V \in C[R_+ \times R^n, R[\Omega, R^m]]$, and define

$$D^+_{(2.1.1)}V(s,x(t,s,y),\omega) \equiv$$
$$\limsup_{n\to 0^+}\frac{1}{h}\Big[V\Big(s+h, x(t,s+h,y+hF(s,y,\omega)),\omega\Big)-V\Big(s,x(t,s,y),\omega\Big)\Big]$$

and

$$D^+V(s,x(t,s,y)-x(t,s,z),\omega) \equiv$$
$$\limsup_{n\to 0^+}\frac{1}{h}\Big[V\Big(s+h, x(t,s+h,y+hF(s,y,\omega))-x(t,s+h,z+h\widehat{F}(s,z)),\omega\Big)$$
$$-V\Big(s,x(t,s,y)-x(t,s,z),\omega\Big)\Big],$$

where $x(t,s,y)$ and $x(t,s,z)$ are solution processes of (2.1.3) through (s,y) and (s,z), respectively. It is clear that $D^+_{(2.1.1)}V(s,x(t,s,y),\omega)$ and $D^+V(s,x(t,s,y)-x(t,s,z),\omega)$ exist for all $(s,y),(s,z) \in R_+ \times R^n$ and are product-measurable random processes.

We shall now formulate the following fundamental comparison theorem in the framework of random vector Lyapunov-like functions.

Theorem 2.2.1. *Suppose that*

(i) $F \in M[R_+ \times R^n, R[\Omega, R^n]]$ *and* F *is smooth enough to guarantee the existence and uniqueness of a sample solution process* $y(t,\omega) = y(t, t_0, y_0(\omega), \omega)$ *of* (2.1.1) *for* $t \geq t_0$;

(ii) $\widehat{F} \in C[R_+ \times R^n, R^n]$, *the solution process* $x(t,\omega) = x(t, t_0, x_0(\omega))$ *of* (2.1.3) *exists for all* $t \geq t_0$, *unique and sample continuous with respect to the initial data and it is locally Lipschitzian in* x_0 *w.p. 1;*

(iii) $V \in C[R_+ \times R^n, R[\Omega, R^m]]$, $V(t,x,\omega)$ *is locally Lipschitzian in* x *w.p. 1, and for* $t_0 \leq s \leq t$, $(s,y) \in R_+ \times R^n$,

$$D^+_{(2.1.1)} V\Big(s, x(t,s,y), \omega\Big) \leq g(s, V(s, x(t,s,y), \omega), \omega), \quad (2.2.1)$$

where $g \in M[R_+ \times R^m, R[\Omega, R^m]]$ *and* $g(t,u,\omega)$ *is sample continuous and quasimonotone nondecreasing in* u *w.p. 1 for fixed* t;

(iv) $r(s,\omega) = r(s, t_0, u_0, \omega)$ *is the maximal sample solution process of the random differential system*

$$u' = g(s, u, \omega), \qquad u(t_0, \omega) = u_0(\omega) \qquad (2.2.2)$$

existing for $s \geq t_0$;

(v) $y(t,\omega) = y(t, t_0, y_0(\omega), \omega)$ *and* $x(t, t_0, y_0(\omega))$ *with* $x_0(\omega) = y_0(\omega)$ *are solution processes of* (2.1.1) *and* (2.1.3) *respectively, and* $x(t,\omega)$ *satisfies*

$$V(t_0, x(t,\omega), \omega) \leq u_0(\omega). \qquad (2.2.3)$$

Then

$$V(t, y(t,\omega), \omega) \leq r(t, t_0, u_0(\omega), \omega), \quad \text{for } t \geq t_0. \qquad (2.2.4)$$

Proof. Set for $t_0 \le s < t$

$$
\begin{aligned}
w(s,\omega) &= V(s, x(t,s,y(s,\omega)),\omega),\\
w(t_0,\omega) &= V(t_0, x(t,\omega),\omega).
\end{aligned}
\tag{2.2.5}
$$

Since $x(t,s,y(s,\omega))$ and $y(s,\omega)$ are sample solution processes of (2.1.3) and (2.1.1) through $(s, y(s,\omega))$ and $(t_0, y_0(\omega))$, respectively, and $V \in C[R_+ \times R^n, R[\Omega, R^m]]$ and the solution process $x(t,s,y)$ of (2.1.3) through (s,y) is continuous with respect to (s,y), it is obvious that $w(s,\omega)$ is sample continuous for $s \ge t_0$. For small $h > 0$ such that $s + h < t$, we have

$$
\begin{aligned}
&w(s+h,\omega) - w(s,\omega)\\
&= V\Big(s+h, x(t,s+h,y(s+h,\omega)),\omega\Big) - V\Big(s, x(t,s,y(s,\omega)),\omega\Big)\\
&= V\Big(s+h, x(t,s+h,y(s+h,\omega)),\omega\Big) - V\Big(s+h, x(t,s+h,y(s,\omega)\\
&\qquad + hF(s,y(s,\omega),\omega)),\omega\Big) + V\Big(s+h, x(t,s+h,y(s,\omega)\\
&\qquad + hF(s,y(s,\omega),\omega)),\omega\Big) - V\big(s, x(t,s,y(s,\omega)),\omega\big).
\end{aligned}
$$

This together with the local Lipschitzian property of $x(t,s,y)$ and $V(t,x,\omega)$ in y and x, respectively, one gets

$$
\begin{aligned}
w(s+h,\omega) - w(s,\omega) \le{}& K\|y(s+h,\omega) - y(s,\omega) - hF(s,y(s,\omega),\omega)\|e\\
&+ V\Big(s+h, x(t,s+h,y(s,\omega) + hF(s,y(s,\omega),\omega)),\omega\Big)\\
&- V\Big(s, x(t,s,y(s,\omega)),\omega\Big),
\end{aligned}
$$

where K is the local Lipschitz constant relative to $x(t,s,y)$ and $V(t,x,\omega)$ and $e = (1,1,\ldots,1)^T \in R^m$. This together with the definition of $D^+_{(2.1.1)}V(s,x(t,s,y),\omega)$, (2.2.1), (2.2.3), (2.2.5), and the sample continuity of $w(s,\omega)$, yields the inequalities

$$
D^+w(s,\omega) \le g(s, w(s,\omega),\omega), \qquad t > s \ge t_0,
$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

and

$$w(t_0, \omega) \le u_0(\omega).$$

Hence by Theorem A.2.4, we have

$$w(s, \omega) \le r(s, t_0, u_0(\omega), \omega), \qquad t_0 \le s < t. \tag{2.2.6}$$

Since $w(s, \omega) \to V(t, x(t, t, y(t, \omega)), \omega) = V(t, y(t, \omega), \omega)$ as $s \to t^-$. This together with (2.2.6) yields

$$V(t, y(t, \omega), \omega) \le r(t, t_0, u_0(\omega), \omega), \text{ w.p. } 1, \qquad t \ge t_0.$$

The proof is complete.

To demonstrate the scope of Theorem 2.2.1, a special case is stated in the following corollary.

Corollary 2.2.1. *Let the hypotheses of* Theorem 2.2.1 *be satisfied except* $\widehat{F}$ *in (ii) is replaced by* $\widehat{F}(t, x) \equiv 0$. *Then* (2.2.1), (2.2.3) *and* (2.2.4) *reduce to*

$$D^+_{(2.1.1)} V(s, y, \omega) \le g(x, V(s, y, \omega), \omega),$$

$$V(t_0, y_0, \omega) \le u_0(\omega)$$

and

$$V(t, y(t, \omega), \omega) \le r(t, t_0, u_0(\omega), \omega) \qquad \textit{for } t \ge t_0,$$

respectively.

This is a well-known (Ladde [59]) comparison theorem for (2.1.1). The following remark will further shed the light on Theorem 2.2.1.

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

Remark 2.2.1. If $u_0 = V(t_0, x(t, t_0, y_0(\omega)), \omega)$, then (2.2.4) becomes

$$V(t, y(t,\omega), \omega) \le r(t, t_0, V(t_0, x(t,\omega), \omega), \omega), \qquad t \ge t_0. \tag{2.2.7}$$

From this it is obvious that the comparison Theorem 2.2.1 is not exactly like the earlier comparison theorems (Ladde [59]). As we see from (2.2.7), this theorem relates the solution processes of three kinds of initial value problems, namely (2.1.1), (2.1.3), and (2.2.2). On the other hand the usual comparison theorems (Ladde [59]) relate the solutions of two kinds of initial value problems. Another difference is the fact that the initial state u_0 of the maximal sample solution process $r(s, t_0, u_0)$ of (2.2.2) depends upon the parameter t.

In the following, we shall give a couple of examples to illustrate the scope of Theorem 2.2.1.

Example 2.2.1. Consider the scalar stochastic differential equation

$$y' = -\frac{1}{2}\, y^3 + H(t, y, \omega), \qquad y(t_0, \omega) = y_0(\omega) \tag{2.2.8}$$

where $H \in M[R_+ \times R, R[\Omega, R]]$, and it is smooth enough to insure a sample solution process of (2.2.8). Further assume that

$$2yH(t, y, \omega) \le \lambda(t, \omega)|y|^2, \qquad \text{for } (t, y) \in R_+ \times R, \tag{2.2.9}$$

where $\lambda \in M[R_+, R[\Omega, R]]$, and it is sample integrable stochastic process. We consider the initial value problem

$$x' = -\frac{1}{2}\, x^3, \qquad x(t_0, \omega) = y_0(\omega). \tag{2.2.10}$$

Thus $x(t, t_0, x_0(\omega)) = y_0(\omega)/[1 + (t - t_0) y_0^2(\omega)]^{1/2}$ and

$$\Phi(t, t_0, y_0(\omega)) = 1/[1 + (t - t_0) y_0^2(\omega)]^{3/2}.$$

By taking $V(t,x,\omega) = \frac{1}{2}|x|^2$, we compute $D^+_{(2.2.8)}V(s, x(t,s,y(s,\omega)),\omega)$. In this case,

$$\begin{aligned}
D^+_{(2.2.8)}V\big(s, x(t,s,y(s,\omega)),\omega\big) &= \frac{d}{ds}\left(y^2(s,\omega)/2[1+(t-s)y^2(s,\omega)]\right) \\
&= y(s,\omega)H(s,y(s,\omega),\omega)/[1+(t-s)y^2(s,\omega)]^2 \\
&\leq \lambda(s,\omega)\frac{1}{2}\left(y^2(s,\omega)/[1+(t-s)y^2(s,\omega)]\right) \\
&\leq \lambda(s,\omega)V\Big(s, x(t,s,y(s,\omega)),\omega\Big), \qquad t \geq s \geq t_0,
\end{aligned}$$

where $x(t,s,y(s,\omega))$ is the solution of (2.2.10) through $(s, y(s,\omega))$, and $y(s,\omega)$ is a solution process of (2.2.8). The comparison equation is

$$u' = \lambda(s,\omega)u, \quad t_0 \leq s < t, \qquad u(t_0,\omega) = u_0(\omega). \tag{2.2.11}$$

Thus $r(s,t_0,u_0) = u_0 \exp\left[\int_{t_0}^{s} \lambda(u,\omega)du\right]$, where $u_0 = \frac{1}{2}|x(t,t_0,y_0(\omega))|^2$. Therefore, the relation (2.2.4) becomes

$$\frac{1}{2}|y(t,\omega)|^2 \leq \frac{1}{2}|x(t,\omega)|^2 \exp\left[\int_{t_0}^{t} \lambda(u,\omega)du\right]. \tag{2.2.12}$$

The above relation justifies the Remark 2.2.1.

Example 2.2.2. Consider the scalar stochastic differential equation

$$y' = \frac{y}{1+t} + H(t,y,\omega), \qquad y(t_0,\omega) = y_0(\omega) \tag{2.2.13}$$

where H is as defined in (2.2.8) and it satisfies

$$2yH(t,y,\omega) \leq -\alpha(\omega)|y|^2 \qquad \text{for } (t,y) \in R_+ \times R \tag{2.2.14}$$

and random variable $\alpha(\omega)$ is positive with probability one. We consider

$$x' = \frac{x}{1+t}, \qquad x(t_0,\omega) = y_0(\omega). \tag{2.2.15}$$

It is obvious that $x(t, t_0, x_0(\omega)) = y_0(\omega)(1+t)/(1+t_0)$, $\Phi(t, t_0, y_0(\omega)) = (1+t)/(1+t_0)$. We pick $V(t, x, \omega) = \frac{1}{2}|x|^2$. After computation, we get

$$\begin{aligned}
D^+_{(2.2.13)} V(s, x(t, s, y(s,\omega)), \omega) &= \frac{d}{ds}\ \left(y^2(s,\omega)(1+t)^2/2(1+s)^2\right) \\
&= yH(s, y, \omega)(1+t)^2/(1+s)^2 \\
&\leq -\alpha(\omega)\frac{1}{2}\, y^2(s,\omega)(1+t)^2/(1+s)^2 \\
&\leq -\alpha(\omega)V(s, x(t, s, y(\omega)), \omega), \qquad t \geq s \geq t_0,
\end{aligned}$$

where $x(t, s, y(\omega))$ is a solution process of (2.2.15) through $(s, y(s,\omega))$, and $y(s,\omega)$ is a solution process of (2.2.13) through $(t_0, y_0(\omega))$. The comparison equation is

$$u' = -\alpha(\omega)u, \qquad u(t_0, \omega) = u_0(\omega). \tag{2.2.16}$$

Hence, $r(s, t_0, u_0) = u_0 \exp[-\alpha(\omega)(s-t_0)]$ with $u_0 = \frac{1}{2}|x(t, t_0, x_0(\omega))|^2$. Thus the relation (2.2.4) becomes

$$\frac{1}{2}|y(t,\omega)|^2 \leq \frac{1}{2}|x(t,\omega)|^2 \exp[-\alpha(\omega)(t-t_0)]. \tag{2.2.17}$$

Example 2.2.3. We consider the initial value problems (2.1.11) and (2.1.14). Let $V(t, x, \omega) = \|x\|^2$. We compute $D^+_{(2.1.11)}V(s, x(t, s, y), \omega)$. For this purpose we consider

$$\begin{aligned}
V(s+h,&x(t, s+h, y+hA(s,y,\omega)y), \omega) - V(t, x(t,s,y), \omega) \\
&= (\|x(t, s+h, y+hA(s,y,\omega)y)\|^2 - \|x(t,s,y)\|^2) \\
&= \Big(\|x(t, s+h, y+hA(s,y,\omega)y)\| \\
&\qquad - \|x(t,s,y)\|\Big)\Big(\|x(t, s+h, y+hA(s,y,\omega)y)\| \\
&\qquad + \|x(t,s,y)\|\Big).
\end{aligned} \tag{2.2.18}$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

From Taylor's expansion, (2.1.15) and (2.2.16), we have

$$\begin{aligned}
&\|x(t,s+h,y+hA(s,y,\omega)y)\| \\
&\qquad = \|x(t,s,y)+h[\Phi(t,s,y)\Big(A(s,y,\omega)-\widehat{A}(s,y)\Big)y]+o(h)\| \\
&\qquad = \|x(t,s,y)+h[\Phi(t,s,y)\Big(A(s,y,\omega)-\widehat{A}(s,y)\Big) \\
&\qquad\qquad \psi^{-1}(t,s,y)x(t,s,y)]+o(h)\|,
\end{aligned}$$

where

$$\psi(t,s,y)=\int_0^1 \Phi(t,s,uy)du.$$

This together with (2.2.18) yields

$$\begin{aligned}
&D^+_{(2.1.11)}V(s,x(t,s,y),\omega) \\
&\le 2\mu\Big(\Phi(t,s,y)\Big(A(s,y,\omega)-\widehat{A}(s,y)\Big)\psi^{-1}(t,s,y)\Big)V(s,x(t,s,y),\omega),
\end{aligned} \tag{2.2.19}$$

where μ is a logarithmic norm of a matrix defined by

$$\begin{aligned}
&\mu\Big(\Phi(t,s,y)\Big(A(s,y,\omega)-\widehat{A}(s,y)\Big)\psi^{-1}(t,s,y)\Big) \\
&= \limsup_{h\to 0^+}\frac{1}{h}[\|I+h[\Phi(t,s,y)\Big(A(s,y,\omega)-\widehat{A}(s,y)\Big)\psi^{-1}(t,s,y)]\|-1].
\end{aligned}$$

Here we assume that

$$\begin{aligned}
&2\mu\Big(\Phi(t,s,y)\Big(A(s,y,\omega)-\widehat{A}(s,y)\Big)\psi^{-1}(t,s,y)\Big)\le\gamma(s,\omega), \\
&\qquad\qquad y\in R^n,\quad t\ge s\ge t_0,
\end{aligned} \tag{2.2.20}$$

where γ is a sample integrable stochastic process. From (2.2.19) and (2.2.20), we get

$$D^+_{(2.1.11)}V(s,x(t,s,y),\omega)\le\gamma(s,\omega)V(s,x(t,s,\omega)). \tag{2.2.21}$$

Therefore the comparison equation is

$$u'=\gamma(s,\omega)u,\qquad u(t_0,\omega)=u_0(\omega). \tag{2.2.22}$$

From (2.2.21) and (2.2.22) and an application of Theorem 2.2.1 in the context of Remark 2.2.1, we obtain

$$V(t, y(t,\omega), \omega) \leq V(t_0, x(t,\omega), \omega) \exp\left[\int_{t_0}^{t} \gamma(s,\omega) ds\right] \qquad (2.2.23)$$

which implies that

$$\|y(t,\omega)\|^2 \leq \|x(t,\omega)\|^2 \exp\left[\int_{t_0}^{t} \gamma(s,\omega) ds\right], \qquad (2.2.24)$$

where $y(t,\omega)$ and $x(t,\omega)$ are solution processes of (2.1.11) and (2.1.14), respectively.

Remark 2.2.2. By adding and subtracting $E[A(s,y,\omega)]$ we can express the right hand side of (2.2.19) as functions of $A(s,y,\omega) - E[A(s,y,\omega)]$ and $E[A(s,y,\omega)] - \widehat{A}(s,y)$. This will be useful in relating to the statistical properties of coefficient rate matrices with the solution processes of (2.1.11), (2.1.13) and (2.1.14). In the light of this and one of the properties of logarithmic norm (Lemma A.2.3), (2.2.19) reduces to

$$\begin{aligned} D^{+}_{(2.1.11)} V(s, x(t,s,y), \omega) \leq \\ 2\Big(\mu\Big(\Phi(t,s,y)\Big(A(s,y,\omega) - E[A(s,y,\omega)]\Big)\psi^{-1}(t,s,y)\Big) \\ +\mu\Big(\Phi(t,s,y)\Big(E[A(s,y,\omega)] - \widehat{A}(s,y)\Big)\psi^{-1}(t,s,y)\Big)\Big) \\ V(s, x(t,s,y), \omega). \quad (2.2.25) \end{aligned}$$

Further, we assume that for $y \in R^n$, $t \geq s \geq 0$,

$$2\mu\Big(\Phi(t,s,y)\Big(A(s,y,\omega) - E[A(s,y,\omega)]\Big)\psi^{-1}(t,s,y)\Big) \leq \alpha(s,\omega) \qquad (2.2.26)$$

and

$$2\mu\Big(\Phi(t,s,y)(E[A(s,y,\omega)] - \widehat{A}(s,y))\psi^{-1}(t,s,y)\Big) \leq \beta(s,\omega), \quad (2.2.27)$$

where α and β are sample integrable stochastic processes. From (2.2.26) and (2.2.27), (2.2.25) reduces to

$$D^+_{(2.1.11)}V(s, x(t, s, y), \omega) \leq \Big(\alpha(s, \omega) + \beta(s, \omega)\Big)V(s, x(t, s, y), \omega). \tag{2.2.28}$$

By following the analysis of Example 2.2.3, we arrive at

$$V(t, y(t, \omega), \omega) \leq V(t_0, x(t, \omega), \omega) \exp\left[\int_{t_0}^{t}(\alpha(s, \omega) + \beta(s, \omega))ds\right]$$

which implies that

$$\|y(t, \omega)\|^2 \leq \|x(t, \omega)\|^2 \exp\left[\int_{t_0}^{t}(\alpha(s, \omega) + \beta(s, \omega))ds\right]. \tag{2.2.29}$$

Example 2.2.4. With regard to systems (2.1.18) and (2.1.20), by following the argument used in Example 2.2.3, we obtain

$$D^+_{(2.1.18)}V(s, x(t, s, y), \omega) \leq$$
$$2\mu\Big(\Phi(t, s)(A(s, \omega) - \widehat{A}(s))\Phi(s, t)\Big)V(s, x(t, s, y), \omega), \tag{2.2.30}$$

which implies that

$$\|y(t, \omega)\|^2 \leq \|x(t, \omega)\|^2 \exp\left[2\int_{t_0}^{t}\mu(\Phi(t, s)(A(s, \omega) - \widehat{A}(s))\Phi(s, t))ds\right]. \tag{2.2.31}$$

We now state and prove a comparison theorem which has wide range of applications in the theory of error estimates and relative stability analysis of systems of stochastic differential equations.

Theorem 2.2.2. *Assume that all the hypotheses of* Theorem 2.2.1 *are satisfied except that relations* (2.2.1) *and* (2.2.3) *are replaced by*

$$D^+V(s, x(t, s, y) - x(t, s, z), \omega) \leq g(s, V(s, x(t, s, y) - x(t, s, z), \omega), \omega) \tag{2.2.32}$$

and

$$V(t_0, x(t,\omega) - \bar{x}(t), \omega) \leq u_0(\omega), \tag{2.2.33}$$

where $x(t, t_0, z_0) = \bar{x}(t)$ *is a solution process of either* (2.1.2) *or* (2.1.3) *depending on the choice of* z_0. *Then*

$$V(t, y(t,\omega) - \bar{x}(t), \omega) \leq r(t, t_0, u_0(\omega), \omega), \qquad t \geq t_0. \tag{2.2.34}$$

Proof. Let $y(t,\omega)$ be a solution process of (2.1.1) and let $x(t,\omega)$ be the solution process of (2.1.3) with $x(t_0,\omega) = y_0(\omega)$. Let $\bar{x}(t) = x(t, t_0, z_0)$ be a solution process of either (2.1.2) or (2.1.3) depending on the choice of z_0. Set for $t_0 \leq s < t$

$$\begin{aligned} w(s,\omega) &= V(s, x(t,s,y(s,\omega)) - x(t,s,\bar{x}(s)), \omega), \\ w(t_0,\omega) &= V(t_0, x(t,\omega) - \bar{x}(t), \omega). \end{aligned}$$

By following the proof of Theorem 2.2.1, the proof of theorem can be completed analogously. Details are left to the reader.

A variation of comparison Theorem 2.2.2 in the context of Remark 2.1.1 and Problem 2.1.3 can be formulated analogously. For the sake of completeness, we state as a corollary.

Corollary 2.2.2. *Assume that all hypotheses of* Theorem 2.2.2 *are satisfied in the context of* $V(s, x(t,s,y-x), \omega)$. *Then the conclusion of* Theorem 2.2.2 *remains valid, provided*

$$V(t_0, x(t, t_0, y_0(\omega) - z_0), \omega) \leq u_0(\omega).$$

To demonstrate the scope of the result, we present a few examples.

Example 2.2.5. We consider the initial value problems (2.1.11), (2.1.13) and (2.1.14). Let $V(t,x,\omega) = \|x\|$. By following the discussion in Example 2.2.3, we can compute $D^+\|x(t,s,y) - x(t,s,z)\|$ as

$$\begin{aligned} D^+\|x(t,s,y) - x(t,s,z)\| \le \mu\Big(\Phi(t,s,y)\Big(A(s,y,\omega) \\ - \widehat{A}(s,y)\Big)\psi^{-1}(t,s,y,z)\Big)\|x(t,s,y) - x(t,s,z)\| \\ + \|\Phi(t,s,y)\Big(A(s,y,\omega) - \widehat{A}(s,y)\Big)\|\|z\|. \end{aligned}$$

If we assume (2.2.20) and

$$\|\Phi(t,s,y)(A(s,y,\omega) - \widehat{A}(s,y))\| \le \nu(s,\omega), \qquad y \in R^n, \quad t \ge s \ge 0 \tag{2.2.35}$$

then the above relation in the context of V reduces to

$$\begin{aligned} D^+V(s, x(t,s,y) - x(t,s,z), \omega) \le \frac{1}{2}\gamma(s,\omega)V\Big(s, x(t,s,y) \\ - x(t,s,z), \omega\Big) + \nu(s,\omega)\|z\|. \end{aligned} \tag{2.2.36}$$

The comparison equation is

$$u' = \frac{1}{2}\gamma(s,\omega)u + \nu(s,\omega)\|z\|, \qquad u(t_0) = u_0. \tag{2.2.37}$$

By applying Theorem 2.2.2, we conclude that

$$\begin{aligned} \|y(t,\omega) - \bar{x}(t)\| \le \|x(t,\omega) - \bar{x}(t)\| \exp\left[\int_{t_0}^t \frac{1}{2}\gamma(s,\omega)ds\right] \\ + \int_{t_0}^t \nu(s,\omega)\|\bar{x}(s)\| \exp\left[\int_s^t \frac{1}{2}\gamma(u,\omega)du\right] ds. \end{aligned} \tag{2.2.38}$$

Remark 2.2.3. A remark similar to Remark 2.2.2 can be formulated, analogously. We simply present the results; final details will be left as an exercise. The final conclusion is

$$\begin{aligned} \|y(t,\omega) - \bar{x}(t)\| \le \|x(t,\omega) - \bar{x}(t)\| \exp\left[\int_{t_0}^t \frac{1}{2}(\alpha(s,\omega) + \beta(s,\omega))\right] ds \\ + \int_{t_0}^t \|\bar{x}(s)\|(\zeta(s,\omega) + \eta(s,\omega)) \exp\left[\frac{1}{2}\int_s^t (\alpha(u,\omega) + \beta(u,\omega))du\right] ds, \end{aligned} \tag{2.2.39}$$

where

$$\|\Phi(t,s,y)(A(s,y,\omega)-E[A(s,y,\omega)])\| \le \zeta(s,\omega), \tag{2.2.40}$$

and

$$\|\Phi(t,s,y)(E[A(s,y,\omega)]-\widehat{A}(s,y))\| \le \eta(s,\omega). \tag{2.2.41}$$

Example 2.2.6. Consider the initial value problems (2.1.18), (2.1.19) and (2.1.20). Assume that

$$\mu\Big(\Phi(t,s)(A(s,\omega)-E[A(s,\omega)])\Phi(s,t)\Big) \le \alpha_1(s,\omega), \tag{2.2.42}$$

$$\mu(\Phi(t,s)(E[A(s,\omega)]-\widehat{A}(s))\Phi(s,t)) \le \beta_1(s,\omega), \tag{2.2.43}$$

$$\|\Phi(t,s)(A(s,\omega)-E[A(s,\omega)])\| \le \zeta_1(s,\omega), \tag{2.2.44}$$

and

$$\|\Phi(t,s)(E[A(s,\omega)]-\widehat{A}(s))\| \le \eta_1(s,\omega), \tag{2.2.45}$$

for all $t \ge s \ge t_0$. From Examples 2.2.4 and 2.2.5 in the context of $V(t,x,\omega) = \|x\|$, we obtain

$$\begin{aligned} D^+V(s,x(t,s,y(s,\omega)) - x(t,s,\bar{x}(s)),\omega) &\le (\alpha_1(s,\omega) \\ &+ \beta_1(s,\omega))V(s,x(t,s,y(s,\omega)) - x(t,s,\bar{x}(s)),\omega) \\ &+ (\zeta_1(s,\omega)+\eta_1(s,\omega))\|\bar{x}(s)\|. \end{aligned} \tag{2.2.46}$$

The comparison equation is

$$u' = (\alpha_1(s,\omega)+\beta_1(s,\omega))u + (\zeta_1(s,\omega)+\eta_1(s,\omega))\|\bar{x}(s)\|, \qquad u(t_0)=u_0. \tag{2.2.47}$$

By the application of Theorem 2.2.2, we have

$$\begin{aligned} \|y(t,\omega)-\bar{x}(t)\| &\le \|x(t,\omega)-\bar{x}(t)\| \exp\left[\int_{t_0}^{t}(\alpha_1(s,\omega)+\beta_1(s,\omega))\right] ds \\ &+ \int_{t_0}^{t}(\zeta_1(s,\omega)+\eta_1(s,\omega))\|\bar{x}(s)\| \exp\left[\int_{s}^{t}(\alpha_1(u,\omega)+\beta_1(u,\omega))du\right] ds. \end{aligned} \tag{2.2.48}$$

2.3. PROBABILITY DISTRIBUTION METHOD

In this section, we present one of the most important methods of determining the probability distribution of the solution process of (2.1.1). The method is based on the fundamental Liouville's theorem in the theory of dynamic systems. The method does not require the explicit form of the solution process of (2.1.1). This approach transforms the problem of determining probability density function of the solution process of (2.1.1) to the problem of solving an initial value problem involving a first-order partial differential equations. Of course, this method is not applicable to very general system of differential equations with random parameters (2.1.1) but it will be limited to a particular class of systems of differential equations with random parameters of the following type

$$y' = h(t, y, \eta(t)), \qquad y(t_0, \omega) = y_0(\omega) \tag{2.3.1}$$

where

$$\eta(t) = \nu(t, A(\omega)), \tag{2.3.2}$$

and it is a random coefficient process with $A(\omega)$ being a finite-dimensional random parameter vector defined on the complete probability space $(\Omega, \mathcal{F}, P)$. The modeling of biological, engineering and physical phenomena by random processes with finite degrees of randomness has been widely studied.

We rewrite (2.3.1) and (2.3.2) as

$$z' = H(t, z), \qquad z(t_0) = z_0(\omega) \tag{2.3.3}$$

where $z = [y^T, A^T]^T \in R^N$, $n < N$, $z_0(\omega) = [y_0^T(\omega), A^T(\omega)]^T$ and $H(t, z) = [h^T(t, y, \nu(t, A)), 0^T]^T$. We note that the initial value prob-

lem (2.3.1) reduces to a random initial value problem with deterministic systems of differential equations (2.3.3). We assume that y_0 and A are known with joint probability density function (JPDF) $p_0(y_0, a, t_0)$.

We state and prove the Liouville-type theorem.

Theorem 2.3.1. *Assume that the joint probability density function* $p(z,t) \equiv p(y,a,t)$ *of the solution process* $z(t)$ *of* (2.3.3) *exists. Then the density function* $p(y,a,t)$ *satisfies Liouville equation*

$$\frac{\partial p(y,a,t)}{\partial t} + \sum_{i=1}^{n} \frac{\partial (p(y,a,t)h_i(t,y,\nu(t,a)))}{\partial y_i} = 0 \qquad (2.3.4)$$
$$p(y,a,t_0) = p_0(y_0,a,t_0),$$

where h and ν are as described in (2.3.1) *and* (2.3.2).

Proof. We define the joint characteristic function of the solution process $z(t) = z(t,t_0,z_0)$ of (2.3.3) by $\phi(u,t)$, where

$$\phi(u,t) = E\left[\exp\left[i\left(\sum_{j=1}^{N} u_i z_i(t)\right)\right]\right], \qquad i = \sqrt{-1}. \qquad (2.3.5)$$

This is essentially the Fourier transform of the probability density function $p(y,a,t)$, that is,

$$\phi(u,t) = \int_{-\infty}^{\infty} \exp\left[i\left(\sum_{j=1}^{N} u_j z_j\right) p(z,t)\right] dz. \qquad (2.3.6)$$

Equation (2.3.5) can be differentiated with respect to t to yield

$$\frac{\partial \phi}{\partial t} = i \sum_{j=1}^{N} u_j E\left[z_j'(t) \exp\left[i\left(\sum_{j=1}^{N} u_j z_j(t)\right)\right]\right].$$

This together with (2.3.3) gives

$$\frac{\partial \phi}{\partial t} = i \sum_{j=1}^{N} u_j E\left[H_j \exp\left[i\left(\sum_{j=1}^{N} u_j z_j(t)\right)\right]\right]$$

and hence

$$\frac{\partial \phi}{\partial t} = i \sum_{j=1}^{N} u_j \int_{-\infty}^{\infty} \exp\left[i \left(\sum_{j=1}^{N} u_j z_j \right) \right] H_j(z,t) p(z,t) dz.$$

From the relationship between the Fourier transforms of a function and its first derivative, the above equation reduces to

$$\frac{\partial \phi}{\partial t} = - \int_{-\infty}^{\infty} \exp\left[i \left(\sum_{j=1}^{N} u_j z_j \right) \right] \sum_{j=1}^{N} \frac{\partial (H_j(z,t) p(z,t))}{\partial z_j} dz. \quad (2.3.7)$$

By differentiating (2.3.6) with respect to t, we have

$$\frac{\partial \phi}{\partial t} = \int_{-\infty}^{\infty} \exp\left[i \left(\sum_{j=1}^{N} u_j z_j \right) \right] \frac{\partial p(z,t)}{\partial t} dz. \quad (2.3.8)$$

From (2.3.7) and (2.3.8), we conclude that

$$\frac{\partial p(z,t)}{\partial t} = - \sum_{j=1}^{N} \frac{\partial (H_j(z,t) p(z,t))}{\partial z_j}.$$

This together with the definitions of z, $p(z,t)$ and $H(z,t)$ yields

$$\frac{\partial p(y,a,t)}{\partial t} + \sum_{j=1}^{n} \frac{\partial (p(y,a,t) h_j(t,y,\nu(t,a)))}{\partial y_j} = 0.$$

This completes the proof of the theorem.

From Theorem 2.3.1, it is obvious that the problem of determining the joint probability density function of the solution process of (2.3.1) reduces to the problem of solving the initial value problem for first-order partial differential equations with the initial value being the joint probability density function of the initial conditions and the parameters. A formal explicit solution of (2.3.4) can be found by

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

means of its associated Lagrange system. For this purpose, we rewrite (2.3.4) in the following form

$$\begin{aligned}\frac{\partial p(y,a,t)}{\partial t} &+ \left(\sum_{j=1}^{n} \frac{\partial h_j(t,y,\nu(t,a))}{\partial y_j}\right) p(y,a,t) \\ &+ \sum_{j=1}^{n} h_j(t,y,\nu(t,a)) \frac{\partial p(y,a,t)}{\partial y_j} = 0 \\ &p(y,a,t_0) = p_0(y_0,a,t_0).\end{aligned} \tag{2.3.9}$$

The Lagrange system of (2.3.9) is

$$\frac{dt}{1} = \frac{dy_1}{h_1} = \frac{dy_2}{h_2} = \cdots = \frac{dy_n}{h_n} = -\frac{dp}{p(\nabla \cdot h)} \tag{2.3.10}$$

where $\nabla \cdot h$ is the divergence of the vector h. System (2.3.10) is equivalent to the following first order system of deterministic ordinary differential equations

$$\begin{aligned} y' &= h(t,y,\nu(t,a)), \qquad y(t_0) = y_0 \\ \frac{dp}{dt} &= -\nabla_y \cdot h(t,y,\nu(t,a))p, \qquad p(y,a,t_0) = p_0(y_0,a,t_0), \end{aligned} \tag{2.3.11}$$

where a is a $N-n$ dimensional deterministic parameter;

$$\begin{aligned}\nabla_y h(t,y,\nu(t,a)) = \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_n}\right) &(h_1(t,y,\nu(t,a)), \\ h_2(t,y,\nu(t,a)), \ldots, &h_n(y,t,\nu(t,a)))^T.\end{aligned}$$

A sufficient condition to determine the solution of system (2.3.11) is given by the following theorem.

Theorem 2.3.2. *Assume that the solution $y(t) = y(t, t_0, y_0, a)$ of the following system*

$$y' = h(t,y,\nu(t,a)), \qquad y(t_0) = y_0$$

has the inverse transform $y_0 = \mu(y, a, t, t_0)$ for all $t \geq t_0$. Then

$$p(y, a, t) = \left(p_0(y_0, a, t_0) \exp\left[-\int_{t_0}^{t} \nabla_y \cdot h(s, y(s), \nu(s, a)) ds \right] \right) \Bigg|_{y_0 = \mu(y, a, t, t_0)} . \tag{2.3.12}$$

Proof. By solving the second equation in (2.3.11), we have

$$p(y, a, t) = C \exp\left[-\int^{t} \nabla_y \cdot h(s, y(s), \nu(s, a)) ds \right] \tag{2.3.13}$$

where C is an arbitrary constant. For $t = t_0$

$$p(y, a, t_0) = p_0(y_0, a, t_0) = C \exp\left[-\int^{t_0} \nabla_y \cdot h(s, y(s), \nu(s, a)) ds \right].$$

From this and (2.3.13), we have

$$p(y, a, t) = \left(p_0(y_0, a, t_0) \exp\left[-\int_{t_0}^{t} \nabla_y \cdot h(s, y(s), \nu(s, a)) ds \right] \right) \Bigg|_{y_0 = \mu(y, a, t, t_0)} .$$

This completes the proof of (2.3.12).

Remark 2.3.1. It is clear that the marginal probability density function can be determined from (2.3.12) by integration. In particular, the probability density function of the solution process of (2.3.1) can be determined by integrating (2.3.12) with respect to a.

Remark 2.3.2. It is interesting to note that the general formula (A.2.20) in Theorem A.2.10 for the determination of density function, in the context of the general formula (2.3.12) in Theorem 2.3.2. Comparing these two formulas, we find that the absolute value of the Jacobian in (A.2.20) can be evaluated by

$$|J(\mu(y, a, t, t_0))| = \exp\left[-\int_{t_0}^{t} \nabla_y \cdot h(s, y(s), \nu(s, a)) ds \right] \Bigg|_{y_0 = \mu(y, a, t, t_0)} .$$

Example 2.3.1. We consider a simple oscillating system without damping as follows:

$$x'' + \lambda^2 x = 0, \qquad x(t_0) = x_0(\omega), \qquad x'(t_0) = v_0(\omega) \tag{2.3.14}$$

where $x_0(\omega)$, $v_0(\omega)$ and $\lambda^2(\omega)$ are random variables. We assume that the joint probability density function $p_0(x_0, v_0, \lambda^2)$ of the three parameters is given.

We rewrite (2.3.14) in its equivalent system of first order differential equations

$$y' = H(t, z), \qquad z(t_0, \omega) = z_0(\omega) \tag{2.3.15}$$

where $z = [y^T, \lambda^2(\omega)]$, $y = [x, v]^T$, $z_0(\omega) = [y_0^T(\omega), \lambda^2(\omega)]^T$, $y_0(\omega) = [x_0(\omega), v_0(\omega)]^T$, $H(t,z) = [v, -\lambda^2(\omega)x, 0]^T$ and $v = x'$. Let $p(z,t) \equiv p(x, v, \lambda, t)$ be the joint probability density of the solution process of (2.3.15). By the application of Theorem 2.3.1, we have

$$\begin{aligned} &\frac{\partial p}{\partial t} + v\,\frac{\partial p}{\partial x} + \lambda^2 x\,\frac{\partial p}{\partial v} = 0 \\ &p(x, v, \lambda, t_0) = p_0(x_0, v_0, t_0). \end{aligned} \tag{2.3.16}$$

The Lagrange system associated with (2.3.16) reduces to the following system of first order ordinary differential equations with deterministic parameter λ^2.

$$\begin{aligned} &\frac{dy}{dt} = h(t, y, \lambda^2), \qquad y(t_0) = y_0 \\ &\frac{dp}{dt} = 0, \qquad p(x, v, \lambda^2, t_0) = p_0(x, v, \lambda^2), \end{aligned} \tag{2.3.17}$$

where $h(t, y, \lambda^2) = [v, -\lambda^2 x]^T$ and $y_0 = [x_0, v_0]^T$. By solving this system, we obtain

$$\begin{aligned} p(x, v, \lambda^2, t) &= C(x, v, \lambda^2) \\ y(t, t_0, y_0) &= [x_0 \cos\lambda(t - t_0) + \frac{v_0}{\lambda}\sin\lambda(t - t_0), \\ &\qquad - \lambda x_0 \sin(t - t_0) + v_0 \cos(t - t_0)]^T, \end{aligned}$$

where C is an arbitrary constant with respect to t and it depends on (x, v, λ^2); the x_0, v_0 and λ are deterministic real numbers. In this case, $y(t, t_0, y_0)$ has an inverse transform

$$\begin{aligned} y_0 &= \mu(y, \lambda, t, t_0) \\ &= [x \cos \lambda(t_0 - t) + \frac{v}{\lambda} \sin(t_0 - t), -\lambda x \sin \lambda(t_0 - \lambda) + v \cos(t_0 - t)]^T \end{aligned}$$

for all $t \geq t_0$. This together with (2.3.17) and the general expression for $p(x, v, \lambda^2, t)$, we have

$$p(x, v, \lambda^2, t_0) = p_0(x_0, v_0, \lambda^2) = C(x, v, \lambda^2).$$

Hence $p_0 = C$. Thus

$$\begin{aligned} p(x, v, \lambda^2, t) = p_0(x_0, v_0, \lambda^2) &= p_0(x_0, v_0, \lambda^2)|_{[x_0, v_0]^T = \mu(y, \lambda, t)} \\ &= p_0(x \cos \lambda(t_0 - t) + \frac{v}{\lambda} \sin \lambda(t_0 - t), \\ &\qquad -\lambda x \sin \lambda(t_0 - t) + v \cos \lambda(t_0 - t), \lambda^2). \end{aligned} \tag{2.3.18}$$

Finally, the marginal probability densities of the solution process of (2.3.14) can be obtained by integrating (2.3.18) with respect to λ.

Example 2.3.2. Let us consider the well-known Verhulst-Pearl logistic equation in population dynamics

$$\frac{dy}{dt} = y(A(\omega) - B(\omega)y), \qquad y(t_0, \omega) = y_0(\omega) \tag{2.3.19}$$

where A, B and y_0 are real-valued random variables defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Here $A(\omega)$ is the random intrinsic rate of natural increase; $A(\omega)/B(\omega)$ is referred as the carrying capacity of the species. It is assumed that the joint probability density function $p_0(y_0, a, b)$ of y_0, A and B is given. Further assume that y_0,

A and B are positive random variables. We rewrite (2.3.19) with its equivalent form as

$$y' = H(t, z), \qquad z(t_0, \omega) = z_0(\omega) \tag{2.3.20}$$

where $H(t,z) = [y(A(\omega)) - B(\omega)y, 0, 0]^T$, $z = [y, A(\omega), B(\omega)]^T$, $z_0(\omega) = [y_0(\omega), A(\omega), B(\omega)]^T$. Let $p(z,t) \equiv p(y,a,b,t)$ be the joint probability density of the solution process of (2.3.20). By application of Theorem 2.3.1, we have

$$\frac{\partial p}{\partial t} + \frac{\partial (ph)}{\partial y} = 0$$
$$p(y,a,b,t_0) = p_0(y_0,a,b)$$

which is equivalent to

$$\begin{gathered} \frac{\partial p}{\partial t} + y(a - by)\frac{\partial p}{\partial y} + (a - 2by)p = 0 \\ p(y,a,b,t_0) = p_0(y_0,a,b). \end{gathered} \tag{2.3.21}$$

The Lagrange system of (2.3.21) gives rise to the following system of first order ordinary differential equations with deterministic parameters a and b.

$$\begin{gathered} \frac{dy}{dt} = y(a - by), \qquad y(t_0) = y_0 \\ \frac{dp}{dt} = -(a - 2by)p, \qquad p(y,a,b,t_0) = p_0(y_0,a,b). \end{gathered} \tag{2.3.22}$$

By solving this system, we have

$$\begin{gathered} y(t,t_0,y_0) = \frac{a}{b}[1 + \exp[-a(t - t_0)](a/b - y_0)/y_0]^{-1} \\ p(y,a,b,t) = p_0(y_0,a,b)\exp\left[-\int_{t_0}^{t}(a - 2by(s))ds\right]. \end{gathered} \tag{2.3.23}$$

We note that $y(t,t_0,y_0)$ in (2.3.23) has inverse transform

$$y_0 = \mu(y,a,b,t) = \frac{a}{b}[1 + \exp[a(t - t_0)](a/b - y)/y]^{-1}. \tag{2.3.24}$$

From (2.3.23), (2.3.24), and the application of Theorem 2.3.2, we have

$$p(y,a,b,t) = \left(p_0(y_0,a,b)\exp\left[-\int_{t_0}^{t} (a - 2by(s))ds \right] \right)\Bigg|_{y_0=\mu(y,a,b,t)}$$

$$= \exp[-a(t-t_0)]$$

$$\times \left(p_0(y_0,a,b)\left[\frac{(a/b - y_0)/y_0 + \exp[a(t-t_0)]}{(a/b - y_0)/y_0 + 1} \right]^{2a/b} \right)\Bigg|_{y_0=\mu(y,a,b,t)}$$

$$p(y,a,b,t) =$$

$$= \exp[-a(t-t_0)]p_0(a/b[1+\exp[-a(t-t_0)](a/b - y)/y]^{-1}, a, b)$$

$$\times \left[\frac{(a/b - y)/y + 1}{(a/b - y)/y + \exp[-a(t-t_0)]} \right]^{2a/b}$$

$$= \exp[-a(t-t_0)]p_0(ay(by + \exp[-a(t-t_0)](a - by))^{-1}, a, b)$$

$$\times \left(\frac{a}{(a - by) + by\exp[-a(t-t_0)]} \right)^{2a/b}.$$

Remark 2.3.3. The conclusions of Theorems 2.3.1 and 2.3.2 can be utilized for obtaining statistical properties, in particular, mean, variance and higher moments of the solution process of (2.3.1). Furthermore, by knowing these moments of solution processes of (2.3.1) one can analyze qualitative properties of the solution process of (2.3.1) in the sense of moments. In particular, finding a deviation between the mean of solution processes of a system of differential equations with random parameters and the solution processes of the deterministic system of differential equations with corresponding means of the random parameters is an open problem. As a result of this, we do not undertake to apply this method to find such a type of bounds on the absolute mean deviation. However, this section provides a tool to undertake such a problem.

2.4. STABILITY ANALYSIS

Let $y(t,\omega) = y(t, t_0, y_0(\omega), \omega)$ be any solution of (2.1.1) and let

$x(t,\omega) = x(t,t_0,y_0(\omega))$ be the solution of (2.1.3) through $(t_0, y_0(\omega))$. Furthermore, let $\bar{x}(t) = x(t,t_0,z_0)$ be the solution of either (2.1.2) or (2.1.3) depending on the choice of z_0. Without loss in generality, we assume that $F(t,0,\omega) \equiv 0$ w.p. 1 and $\widehat{F}(t,0) \equiv 0$ for all $t \geq 0$. $y(t,\omega) \equiv 0$, $x(t,\omega) \equiv 0$, $\bar{x}(t) \equiv 0$ are the unique solutions of underlying initial value problems.

Definition 2.4.1. The trivial solution process of (2.1.1) is said to be (SM$_1$) *stable in the p^{th} moment*, if for each $\epsilon > 0$, $t_0 \in R_+$ and $p \geq 1$, there exist a positive function $\delta(t_0,\epsilon)$ such that the inequality $||y_0||_p \leq \delta$ implies

$$||y(t)||_p < \epsilon, \qquad t \leq t_0$$

where

$$||y(t)||_p = \left(\int_{\Omega} ||y(t,\omega)||^p P(d\omega) \right)^{1/p};$$

(SM$_2$) *asymptotically stable in the p^{th} moment*, if it is stable in the p^{th} moment and if for any $\epsilon > 0$, $t_0 \in R$, there exists $\delta_0(t_0)$ and $T = T(t_0,\epsilon)$ such that the inequality $||y_0||_p \leq \delta_0$ implies

$$||y(t)||_p < \epsilon, \qquad t \geq t_0 + T.$$

Remark 2.4.1. We note that depending on the mode of convergence in the probabilistic analysis, one can formulate other definitions of stability and boundedness. See Ladde and Lakshmikantham [67].

For the sake of easy reference, we formulate a concept of relative stability.

Definition 2.4.2. The two differential systems (2.1.1) and (2.1.2) are said to be

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

(RM_1) *relatively stable in p^{th} moment*, if for each $\epsilon > 0$, $t_0 \in R_+$, and $p \geq 1$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ such that the inequality $\|y_0 - m_0\|_p \leq \delta$ implies

$$\|y(t) - m(t)\|_p < \epsilon, \qquad t \geq t_0;$$

(RM_2) *relatively asymptotically stable in the p^{th} moment*, if it is relatively stable in the p^{th} moment and if for any $\epsilon > 0$, $t_0 \in R_+$, there exist $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that the inequality $\|y_0 - m_0\|_p \leq \delta_0$ implies

$$\|y(t) - m(t)\|_p < \epsilon, \qquad t \geq t_0 + T.$$

Remark 2.4.2. Based on Definitions 2.4.1 and 2.4.2, a definition relative to (2.1.1) and (2.1.3) can be formulated analogously.

The use of the comparison method to study the stability analysis of (2.1.1) requires the information about the stability of comparison differential system (2.2.2). For example, the stability concepts in Definition 2.4.1 relative to (2.2.2) will be denoted by (SM_1^*) and (SM_2^*). In the present framework, we need a joint stability property of (2.2.2) and (2.1.2), or (2.2.2) and (2.1.3). This does not necessarily imply that each of the systems (2.2.2) and (2.1.2) (or (2.2.2) and (2.1.3)) possess the same kind of stability property. In fact, it is this ingredient that offers more flexibility in applications than the classical approach (Ladde [59]).

Let $x(t, \omega) = x(t, t_0, y_0(\omega))$ and $u(t, t_0, u_0)$ be solutions of (2.1.3) and (2.2.2) through $(t_0, y_0(\omega))$ and (t_0, u_0) respectively. Then we define

$$\nu(t, t_0, y_0(\omega), \omega) = u(t, t_0, V(t_0, x(t, t_0, y_0(\omega)), \omega)) \qquad (2.4.1)$$

and note that $\nu(t_0, t_0, y_0(\omega), \omega) = V(t_0, y_0(\omega), \omega)$ and V is as defined in Theorem 2.2.1. We now formulate joint stability concepts relative (2.1.3) and (2.2.2) with $x_0(\omega) = y_0(\omega)$.

Definition 2.4.3. The trivial solution processes $x \equiv 0$ and $u \equiv 0$ of (2.1.3) and (2.2.2) are said to be

(JM_1) *jointly stable in the mean,* if for $\epsilon > 0$, $t_0 \in R_+$, there exists a $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that $\sum_{i=1}^{m} E\Big[\nu_i(t_0, t_0, y_0(\omega), \omega)\Big] \leq \delta_1$ implies

$$\sum_{i=1}^{m} E[\nu_i(t, t_0, y_0(\omega), \omega)] < \epsilon, \qquad t \geq t_0;$$

(JM_2) *jointly asymptotically stable in the mean,* if it is jointly stable in the mean and if for any $\epsilon > 0$, $t_0 \in R_+$, there exist $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \epsilon) > 0$ such that the inequality $\sum_{i=1}^{m} E\Big[\nu_i(t_0, t_0, y_0(\omega), \omega)\Big] \leq \delta_1$ implies

$$\sum_{i=1}^{m} E[\nu_i(t, t_0, y_0(\omega), \omega)] < \epsilon, \qquad t \geq t_0 + T.$$

The joint relative stability of (2.2.2) and (2.1.2) (or (2.2.2) and (2.1.3)) is defined as follows:

Definition 2.4.4. The system (2.2.2) and (2.1.2) (or (2.1.3)) are said to be (JR_1) *jointly relatively stable in the mean,* if for each $\epsilon > 0$, $t_0 \in R_+$, there exists $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that the inequality $\sum_{i=1}^{m} E\Big[\nu_i(t_0, t_0, y_0(\omega) - z_0, \omega)\Big] \leq \delta_1$ implies

$$\sum_{i=1}^{m} E\Big[\nu_i(t, t_0, y_0(\omega) - z_0, \omega)\Big] < \epsilon, \qquad t \geq t_0;$$

whenever $\|z_0\|$ is small enough.

(JR_2) *jointly relatively asymptotically stable in the mean*, if it is jointly relatively stable in the mean and if for any $\epsilon > 0$, $t_0 \in R_+$, there exists a $\delta^0(t_0) = \delta^0 > 0$ and $T^0 = T^0(t_0, \epsilon) > 0$ such that $\sum_{i=1}^{m} E[\nu_i(t_0, t_0, y_0(\omega) - z_0, \omega)] < \delta^0$ implies

$$\sum_{i=1}^{n} E[\nu_i(t, t_0, y_0(\omega) - z_0, \omega)] < \epsilon, \qquad t > t_0.$$

Remark 2.4.3. We remark that the above notions of joint stability relative to (2.2.2) and (2.1.3)(or (2.2.2) and (2.1.2)) reduce to the stability in the mean of the trivial solution of our comparison system (2.2.2).

We shall present a more general stability criterion that assures the stability in the p^{th} moment of the trivial solution processes of (2.1.1). Furthermore, some illustrations are given to show that the stability conditions are connected with the statistical properties of random rate functions in the differential equations. Examples are worked out to exhibit the advantages of the joint stability concepts.

Theorem 2.4.1. *Let the hypotheses of* Theorem 2.2.1 *be satisfied. Further assume that* $\widehat{F}(t, 0) \equiv 0$, $F(t, 0, \omega) \equiv 0$ *and* $g(t, 0, \omega) \equiv 0$ *with probability one, and for* $(t, x) \in R_+ \times R^n$,

$$b(\|x\|^p) \leq \sum_{i=1}^{m} V_i(t, x, \omega) \leq a(t, \|x\|^p) \tag{2.4.2}$$

where $b \in \mathcal{VK}$, $a \in \mathcal{CK}$ *and* $p \geq 1$. *Then*

(JM_1) *of* (2.2.2) *and* (2.1.3) *implies* (SM_1) *of* (2.1.1);

and

(JM_2) of (2.2.2) and (2.1.3) implies (SM_2) of (2.1.1).

Proof. Let $\epsilon > 0$, $t_0 \in R_+$ be given. Assume that (JM_1) holds. Then for $b(\epsilon) > 0$ and $t_0 \in R_+$, there exists a $\delta_1(\epsilon) = \delta_1(\epsilon, t_0)$ such that $\sum_{i=1}^{m} E\Big[\nu_i(t_0, t_0, y_0(\omega), \omega)\Big] \leq \delta_1(\epsilon)$ implies

$$\sum_{i=1}^{m} E[\nu_i(t, t_0, y_0(\omega), \omega)] < b(\epsilon^p), \qquad t \geq t_0 \tag{2.4.3}$$

where

$$\nu(t, t_0, y_0(\omega), \omega) = r(t, t_0, V(t_0, x(t, t_0, y_0(\omega)), \omega), \omega), \tag{2.4.4}$$

$r(t, t_0, u_0, \omega)$ is the maximal solution process of (2.2.2) and $x(t, t_0, y_0(\omega))$ is the solution process of (2.1.3) through $(t_0, y_0(\omega))$. For $\delta_1(\epsilon)$, we find $\delta(\epsilon)$ so that $a\big(t_0, E\big[\|y_0\|^p\big]\big) < \delta_1(\epsilon)$ whenever $\|y_0\|_p \leq \delta(\epsilon)$. Now we claim that if $\|y_0\|_p \leq \delta$ implies $\|y(t)\|_p < \epsilon$, $t \geq t_0$. Suppose that this is false. Then there would exist a solution process $y(t, t_0, y_0(\omega), \omega)$ with $\|y_0\|_p \leq \delta$ and a $t_1 > t_0$ such that

$$\|y(t_1)\|_p = \epsilon \text{ and } \|y(t)\|_p \leq \epsilon, \qquad t \in [t_0, t_1]. \tag{2.4.5}$$

On the other hand, by Theorem 2.2.1 in the context of Remark 2.2.1, we have

$$V(t, y(t, \omega), \omega) \leq r(t, t_0, V(t_0, x(t, \omega), \omega), \omega), \qquad t \geq t_0. \tag{2.4.6}$$

From (2.4.2) and using the convexity of b, we obtain

$$\begin{aligned} b(E[\|y(t)\|^p]) &\leq \sum_{i=1}^{m} E[V_i(t, y(t, \omega), \omega)] \\ &\leq \sum_{i=1}^{m} E[V_i(t, t_0, y_0(\omega), \omega)], \qquad t \geq t_0. \end{aligned} \tag{2.4.7}$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Relations (2.4.3), (2.4.5), (2.4.6) and (2.4.7) lead to the contradiction

$$\begin{aligned} b(\epsilon^p) &\le \sum_{i=1}^{m} E[V_i(t_1, y(t_1,\omega),\omega)] \\ &\le \sum_{i=1}^{m} E[V_i(t_1, t_0, y_0(\omega),\omega)] < b(\epsilon^p), \end{aligned} \tag{2.4.8}$$

which proves (SM_1). The proof of (SM_2) can be formulated, analogously. The proof of the Theorem is complete.

The following examples illustrate the scope and usefulness of joint stability concept and Theorem 2.4.1.

Example 2.4.1. Let us consider Example 2.2.1. We assume that $H(t, 0, \omega) \equiv 0$ w.p. 1. Furthermore, λ in (2.2.9) satisfies

$$E\left[\exp\left[\int_{t_0}^{t} \lambda(s,\omega)ds\right]\right] \le M$$

for some positive number M, and it is independent of $y_0(\omega)$ with $\|y_0\|_2 < \infty$. It is clear that

$$\nu(t, t_0, y_0(\omega), \omega) = \frac{1}{2}|x(t,\omega)|^2 \exp\left[\int_{t_0}^{t} \lambda(s,\omega)ds\right].$$

This together with the assumptions about H, $y_0(\omega)$, $\lambda(t,\omega)$ and the dominated convergence theorem (Theorem A.2.9), it follows that the trivial solution processes $x \equiv 0$ and $u \equiv 0$ of (2.2.10) and (2.2.11) are jointly stable in the mean. Moreover, from the conditions on λ, y_0, and the nature of the solution process $x(t,\omega)$, one can conclude that $x \equiv 0$ and $u \equiv 0$ of (2.2.10) and (2.2.11) are jointly asymptotically stable in the mean square. From this and the application of Theorem 2.4.1, one can conclude that the trivial solution process of (2.2.8) is asymptotically stable in the 2-nd moment.

Example 2.4.2. Consider Example 2.2.2 except the relation (2.2.14) is replaced by

$$2yH(t,y,\omega) \le (\bar{\alpha} + \eta(t,\omega))|y|^2 \tag{2.4.9}$$

where $\bar{\alpha} \in R$; $\eta(t,\omega)$ is a stationary Gaussian process with mean $E[\eta(t,\omega)] = 0$, and the covariant function $C(t-s) = E[\eta(t,\omega)\eta(s,\omega)]$. $H(t,0,\omega) \equiv 0$ w.p. 1. By following the discussion in Example 2.2.2, we have

$$\begin{aligned} D_{(2.2.13)}&V(s,x(t,s,y(s,\omega)),\omega) \\ &\le (\bar{\alpha} + \eta(s,\omega))V(s,x(t,s,y(s,\omega)),\omega), \qquad t > s \ge t_0, \\ u' &= (\bar{\alpha} + \eta(s,\omega))u, \qquad u(t_0,\omega) = u_0(\omega), \end{aligned} \tag{2.4.10}$$

where

$$V(t,x,\omega) = \frac{1}{2}|x|^2,$$

and

$$V(t,y(t,\omega),\omega) \le V(t_0,x(t,\omega),\omega)\exp\left[\int_{t_0}^{t}(\bar{\alpha} + \eta(s,\omega))ds\right], \quad t \ge t_0. \tag{2.4.11}$$

From the properties of Gaussian process, it is obvious that

$$E\left[\exp\left[\int_{t_0}^{t}(\bar{\alpha} + \eta(s,\omega))ds\right]\right] = \exp\left[\bar{\alpha}(t-t_0) + \frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{t}C(u-s)du\,ds\right].$$

The trivial solutions of (2.2.15) and (2.4.10) are jointly stable in the mean if, $\|y_0\|_2 < \infty$, y_0 and η are independent, $\eta(t,\omega)$ has bounded spectral density $d(\lambda)$ that satisfies

$$2\bar{\alpha} + d(0) < 0. \tag{2.4.12}$$

From (2.4.10) and (2.4.11), we can conclude that the trivial solution process of (2.2.13) is stable in mean-square. Moreover, it is asymptotically stable in the 2-nd moment.

Remark 2.4.4. Example 2.4.2 shows that the trivial solution process $y \equiv 0$ of (2.2.13) is asymptotically stable in the mean-square. We note that the trivial solution process $x \equiv 0$ of (2.2.15) is unstable in the mean square. In fact, $E[|x(t,\omega)|^2] \to \infty$ as $t \to \infty$ provided $E[|y_0|^2] < \infty$. The presented method (Theorem 2.2.1) provides a tool to incorporate the perturbation effects characterized by (2.4.9). Thus the joint stability motivated by Theorem 2.2.1 provides greater advantage to study the stability problem.

Problem 2.4.1. By considering Example 2.2.3, give sufficient conditions for the mean-square stability of the trivial solution of (2.1.11).

In the following, we shall illustrate the use of the variation of constants method to study the p^{th} moment stability of the trivial solution processes of (2.1.1).

Theorem 2.4.2. *Let the hypotheses of* Theorem 2.1.1 *be satisfied. Furthermore, assume that* V, $x(t,\omega)$, $\Phi(t,s,y)$, $R(s,y,\omega)$ *and* $\widehat{F}(t,y)$ *satisfy*

(C_1) $b(\|x\|^p) \le \|V(t,x,\omega)\| \le a(\|x\|^p)$ *for all* $(t,x) \in R_+ \times R^n$, *where* $b \in \mathcal{VK}$ *and* $a \in \mathcal{CK}$;

(C_2) $\widehat{F}(t,0) \equiv 0$ *and* $R(t,0,\omega) \equiv 0$ *with probability one for* $t \in R_+$;

(C_3) $\|\mathcal{D}V(s,x(t,s,y),\omega)\| \le \lambda(s,\omega)V(s,y,\omega)$ *for* $t_0 \le s \le t$, $\|y\|^p \le \rho$, *where* ρ *is some positive real,*

$$\mathcal{D}V(s,x(t,s,y),\omega) = V_s(s,x(t,s,y),\omega) + V_x(s,x(t,s,y),\omega)\Phi(t,s,y)R(s,y,\omega), \quad (2.4.13)$$

and $\lambda \in M[R_+, R[\Omega, R_+]]$ *and it satisfies the relation*

$$E\left[\exp\left[\int_{t_0}^{t} \lambda(s,\omega)ds\right]\right] \le \exp\left[\int_{t_0}^{t} \widehat{\lambda}(s)ds\right], \quad (2.4.14)$$

$\widehat{\lambda} \in L^1[R_+, R_+]$, *$\lambda(t,\omega)$ and $y_0(\omega)$ are independent random variables;*

(C$_4$) $\|V(t_0, x(t,\omega), \omega)\| \le \alpha(\|y_0(\omega)\|^p)$, *whenever* $E[\|y_0\|^p] \le \rho$ *for some* $\rho > 0$ *and where* $\alpha \in \mathcal{CK}$.

Then the trivial solution process of (2.1.1) is stable in the p^{th} mean.

Proof. Let $y(t,\omega)$ be a sample solution process of (2.1.1). Let $x(t,s,y(s,\omega))$ and $x(t,\omega) = x(t,t_0,y_0(\omega))$ be the sample solution processes of (2.1.3) through $(s, y(s,\omega))$ and $(t_0, y_0(\omega))$, respectively, for $t_0 \le s \le t$ and $t_0 \in R_+$. From hypothesis (C$_2$), we have $x(t,t_0,0) \equiv 0$ and $y(t,t_0,0,\omega) \equiv 0$ with probability one. From Theorem 2.1.1, conditions (C$_3$) and (C$_4$), we have

$$\begin{aligned}\|V(t, y(t,\omega), \omega)\| &\le \|V(t_0, x(t,\omega), \omega)\| + \int_{t_0}^{t} \lambda(s,\omega)\|V(s, y(s,\omega), \omega)\|ds \\ &\le \alpha(\|y_0(\omega)\|^p) + \int_{t_0}^{t} \lambda(s)\|V(s, y(s,\omega), \omega)\|ds\end{aligned}$$

as long as $E[\|y(s,\omega)\|^p] \le \rho$. By setting $m(t,\omega) = \|V(t, y(t,\omega), \omega)\|$, the above inequality reduces to

$$m(t,\omega) \le \alpha(\|y_0(\omega)\|^p) + \int_{t_0}^{t} \lambda(s,\omega) m(s,\omega)ds$$

as long as $E[\|y(s,\omega)\|^p] \le \rho$ for $t_0 \le s \le t$. This together with the application of Bellman-Gronwell-Reid inequality Lemma A.2.4, yields

$$m(t,\omega) \le \alpha(\|y_0(\omega)\|^p) \exp\left[\int_{t_0}^{t} \lambda(s,\omega)ds\right],$$

which implies that

$$\|V(t, y(t,\omega), \omega)\| \le \alpha(\|y_0(\omega)\|^p) \exp\left[\int_{t_0}^{t} \lambda(s,\omega)ds\right], \qquad (2.4.15)$$

as long as $E[\|y(s,\omega)\|^p] \leq \rho$ for $t_0 \leq s \leq t$. From the nature of α, $y_0(\omega)$ and λ, (2.4.15) reduces to

$$E[\|V(t, y(t,\omega), \omega)\|] \leq \alpha(E[\|y_0(\omega)\|^p]) \exp\left[\int_{t_0}^{t} \widehat{\lambda}(s)ds\right]$$

as long as $E[\|y(s,\omega)\|^p] \leq \rho$, $t_0 \leq s \leq t$. From this and condition (C_1) and the L^1 property of $\widehat{\lambda}$, we have

$$b(E[\|y(t,\omega)\|^p]) \leq K\alpha(E[\|y_0(\omega)\|^p]), \tag{2.4.16}$$

as long as $E[\|y(t,\omega)\|^p] \leq \rho$, $t \geq t_0$. To conclude $E[\|y(t,\omega)\|^p] \leq \rho$ for $t \geq t_0$ and the (SM_1) stability property of (2.1.1), first we pick $y_0(\omega)$ such that

$$K\alpha(E[\|y_0(\omega)\|^p]) < b(\rho).$$

For this choice of y_0, we claim that $E[\|y(t,\omega)\|^p] < \rho$ for all $t \geq t_0$. If this claim is false, then there exists $\bar{t} \in R_+$ such that $\bar{t} > t_0$, $E[\|y(s,\omega)\|^p] < \rho$, for $t_0 \leq s < \bar{t}$ and $E[\|y(\bar{t},\omega)\|^p] = \rho$. For $t \in [t_0, \bar{t}]$, (2.4.15) is valid. Hence using relation (2.4.16), we get $b(\rho) = b(E[\|y(\bar{t},\omega)\|^p]) < b(\rho)$. This contradiction establishes the claim that $E[\|y(t,\omega)\|^p] < \rho$. Hence (2.4.15) is valid for all $t \geq t_0$. Finally, we need to conclude (SM_1) of (2.1.1). For this purpose, we pick any $\epsilon > 0$ and choose $y_0(\omega)$ such that

$$K\alpha(E[\|y_0(\omega)\|^p]) < b(\epsilon^p). \tag{2.4.17}$$

This implies that

$$(E[\|y_0(\omega)\|^p])^{1/p} < \left(\alpha^{-1}\left(\frac{1}{K}\,b(\epsilon^p)\right)\right)^{1/p}.$$

From (2.4.16) and (2.4.17), we have

$$(E[\|y(t,\omega)\|^p])^{1/p} < \epsilon, \qquad t \geq t_0$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

whenever

$$(E[\|y_0(\omega)\|^p])^{1/p} < \delta(\epsilon),$$

where

$$\delta(\epsilon) = \left(\alpha^{-1}\left(\frac{1}{K}\,b(\epsilon^p)\right)\right)^{1/p} > 0.$$

This is because of the properties of the functions b, α, and α^{-1}. This completes the proof of the theorem.

Next theorem provides sufficient conditions for the p^{th} moment asymptotic stability of the trivial solution of (2.1.1).

Theorem 2.4.3. *Assume that the hypotheses of* Theorem 2.4.2 *hold except that* (C_3) *and* (C_4) *are replaced by*
(C_5):

$$\|\mathcal{D}V(s, x(t,s,y), \omega)\| \le \lambda(s,\omega)\eta(t-s)\|V(s,y,(\omega))\| \tag{2.4.18}$$

for $t_0 \le s \le t$, $\|y\|^p \le \rho$ *for some* $\rho > 0$,
(C_6):

$$\|V(t_0, x(t,\omega), \omega)\| \le \alpha(\|y_0(\omega)\|^p)\beta(t-t_0), \qquad t \ge t_0,$$

provided $E[\|y_0(\omega)\|^p] \le \rho$, *where* λ *and* α *are as defined in* (C_3) *and* (C_4); $\eta, \beta \in \mathcal{L}$, *and satisfy*

$$\eta(t-s)\bar{\eta}(s-t_0) \le k\bar{\eta}(t-t_0) \qquad \text{for some } \bar{\eta} \in \mathcal{L}, \quad k > 0$$

and

$$\lim_{t\to\infty}(\bar{\eta}(t-t_0))E\left[\int_{t_0}^{t} \frac{k\lambda(s,\omega)\beta(s-t_0)}{\bar{\eta}(s-t_0)} \exp\left[\int_s^t k\lambda(u,\omega)du\right] ds\right] = 0. \tag{2.4.19}$$

Then the trivial solution process of (2.1.1) *is asymptotically stable in the* p^{th} *moment.*

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Proof. By employing the conditions (C_5), (C_6), and imitating the proof of Theorem 2.4.2, we arrive at

$$\|V(t,y(t,\omega),\omega)\| \leq \alpha(\|y_0(\omega)\|^p)\beta(t-t_0) + \int_{t_0}^{t} \lambda(s,\omega)\eta(t-s)\|V(s,y(s,\omega),\omega)\|ds \quad (2.4.20)$$

as long as $E[\|y(t,\omega)\|^p] \leq \rho$. By setting

$$m(t,\omega) = \frac{\|V(t,y(t,\omega),\omega)\|}{\bar{\eta}(t-t_0)}, \qquad n(t,\omega) = \frac{\alpha(\|y_0(\omega)\|^p)\beta(t-t_0)}{\bar{\eta}(t-t_0)}$$

and $\nu(s,\omega) = \lambda(s,\omega)k$, (2.4.20) can be rewritten as

$$m(t,\omega) \leq n(t,\omega) + \int_{t_0}^{t} \nu(s,\omega)m(s,\omega)ds$$

as long as $E[\|y(t,\omega)\|^p] \leq \rho$. Theorem A.2.5 readily gives the estimate

$$m(t,\omega) \leq n(t,\omega) + \int_{t_0}^{t} \nu(s,\omega)n(s,\omega)\left(\exp\left[\int_{s}^{t} \nu(u,\omega)du\right]\right)ds. \quad (2.4.21)$$

From the nature of functions η, λ, and the definitions of $\nu(t,s,\omega)$, $m(t,\omega)$ and $n(t,\omega)$, (2.4.21) becomes

$$\|V(t,y(t,\omega),\omega)\| \leq n(t,\omega)\bar{\eta}(t-t_0) + \bar{\eta}(t-t_0)\left[\int_{t_0}^{t} \nu(s,\omega)n(s,\omega)\left(\exp\left[\int_{s}^{t} \nu(u,\omega)du\right]\right)ds\right] \quad (2.4.22)$$

as long as $E[\|y(t,\omega)\|^p] \leq \rho$. From (2.4.22), condition (C_1) and properties of functions α, λ, and the nature of y_0 and λ, we obtain

$$b(E[\|y(t,\omega)\|^p]) \leq \alpha(E[\|y_0(\omega)\|^p])[\beta(t-t_0) + \bar{\eta}(t-t_0)] \; E\left[\int_{t_0}^{t} \frac{\nu(s,\omega)\beta(s-t_0)}{\bar{\eta}(s-t_0)}\left(\exp\left[\int_{s}^{t} \nu(u,\omega)du\right]\right)ds\right] \quad (2.4.23)$$

as long as $E[\|y(t,\omega)\|^p] \leq \rho$. From (2.4.23) and the properties of underlining functions one can easily conclude that $E[\|y(t,\omega)\|^p] \leq \rho$

for all $t \geq t_0$. Moreover, (2.4.20) can be rewritten like (2.4.15). Hence the (SM_1) property of the trivial solution of (2.1.1) can be concluded by following the argument of Theorem 2.4.2. To conclude the (SM_2), it is obvious from (2.4.23), (2.4.19) and the nature of β, $b(E[\|y(t,\omega)\|^p])$ tends to zero as $t \to \infty$. Hence, one can manipulate to verify the technical definition of (SM_2). We leave the details as an exercise. This completes the proof of the theorem.

Remark 2.4.5. We note that Theorems 2.4.1, 2.4.2 and 2.4.3 also provide the sufficient conditions for the stability with probability one of the trivial solution process of (2.1.1). This fact follows from relations (2.4.6), (2.4.15) and (2.4.22). Furthermore, the stability properties in probability follow by the use of Jensen's inequality (Theorem A.1.3).

Remark 2.4.6. The conditions (C_4) and (C_6) in Theorems 2.4.2 and 2.4.3, respectively imply uniform stability and asymptotic stability in the p^{th} moment of the trivial solution process of (2.1.3). This fact can be justified from the deterministic theorems. See Theorems A.2.6 and A.2.7.

To appreciate the assumptions of Theorem 2.4.3, we present the following result which is applicable to many problems. We omit (2.4.19) and modify the condition on η.

Corollary 2.4.1. *Let the hypotheses of* Theorem 2.4.3 *be satisfied except that* (2.4.19) *and the condition on* η *are replaced by*

$$\eta(t-s)\beta(s-t_0) \leq k\beta(t-t_0), \qquad t \geq t_0 \tag{2.4.24}$$

and

$$\lim_{t\to\infty}\left[\beta(t-t_0)E\left[\exp\left[k\int_{t_0}^{t}\lambda(s,\omega)ds\right]\right]\right] = 0, \tag{2.4.25}$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

where k is some positive constant. Then the trivial solution (2.1.1) *is asymptotically stable in the p^{th} moment.*

Proof. By following the proof of Theorem 2.4.3, we arrive at (2.4.20). Now, by using (2.4.24), (2.4.20) can be rewritten as

$$m(t,\omega) \leq \alpha(\|y_0(\omega)\|^p) + \int_{t_0}^{t} k\lambda(s,\omega)m(s,\omega)ds, \tag{2.4.26}$$

as long as $E[\|y(t,\omega)\|^p] \leq \rho$, where

$$m(t,\omega) = \frac{\|V(t,y(t,\omega),\omega)\|}{\beta(t-t_0)}.$$

By applying Lemma A.2.4 to (2.4.26), we get

$$m(t,\omega) \leq \alpha(\|y_0(\omega)\|^p) \exp\left[k\int_{t_0}^{t} \lambda(s,\omega)ds\right]$$

which implies

$$b(\|y(t,\omega)\|^p) \leq \alpha(\|y_0(\omega)\|^p)\beta(t-t_0) \exp\left[k\int_{t_0}^{t} \lambda(s,\omega)ds\right].$$

By taking expectation both sides and using the properties of b, α and $y_0(\omega)$, we have

$$b(E[\|y(t,\omega)\|^p]) \leq \alpha(E[\|y_0(\omega)\|^p])\beta(t-t_0)E\left[\exp\left[k\int_{t_0}^{t} \lambda(s,\omega)ds\right]\right]. \tag{2.4.27}$$

The rest of the proof can be completed by using the argument in the proof of Theorem 2.4.3. The proof of the corollary is complete.

Remark 2.4.7. If $\eta(r) = \beta(r) = \exp[-\gamma r]$ for some $\gamma > 0$, then they are admissible in Corollary 2.4.1.

We discuss a few examples to exhibit the fruitfulness of our results.

Example 2.4.3. Let us consider Example 2.1.1. We assume that the solution process $x(t, t_0, y_0(\omega)) = x(t, \omega) = \Phi(t, s)y_0(\omega)$ fulfills the following relation

$$\|x(t, \omega)\| \leq \|y_0\| \exp[-\gamma(t - t_0)], \qquad t \geq t_0 \tag{2.4.28}$$

for some $\gamma > 0$. Further assume that $\|A(s, \omega) - \widehat{A}(s)\|$ satisfies

$$E\left[\exp\left[2\int_{t_0}^{t} \|A(s, \omega) - \widehat{A}(s)\| ds\right]\right] \leq \exp[\gamma_1(t - t_0)] \tag{2.4.29}$$

for some $\gamma_1 \in R_+$. Let $V(t, x, \omega) = \|x\|^2$. In this case,

$$\mathcal{D}V(s, x(t, s, y(s, \omega), \omega)) = \\ 2y^T(s, \omega)\Phi^T(t, s)\Phi(t, s)[A(s, \omega) - \widehat{A}(s)]y(s, \omega)$$

which satisfied condition (C_5) with $\lambda(s, \omega) = 2\|A(s, \omega) - \widehat{A}(s)\|$, $\eta(t - s) = \exp[-2\gamma(t - s)]$, $p = 2$, $0 < \rho < \infty$. From this and (2.4.28), it is clear that $\beta \equiv \eta$. In the light of Remark 2.4.7, (2.4.28) and (2.4.29), these functions satisfy (2.4.24) and (2.4.25), whenever $\gamma_1 < 2\gamma$. Thus Corollary 2.4.1 is applicable to system (2.1.18) to conclude the asymptotic stability in the mean-square of the trivial solution process of (2.1.18).

We present an example that shows the superiority of Theorem 2.4.1 over Theorem 2.4.3.

Example 2.4.4. Again, consider Example 2.2.1. We assume that (2.2.8) satisfies all the requirements of Examples 2.2.1 and 2.4.1. By applying Theorem 2.1.1 relative to (2.2.8) in the context of $V(t, x, \omega) = \frac{1}{2}|x|^2$, we obtain

$$|y(t, \omega)|^2 = |x(t, \omega)|^2 + 2\int_{t_0}^{t} \frac{(y(s, \omega)H(s, y(s, \omega), \omega))}{[1 + (t - s)y^2(s, \omega)]^2} ds.$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

This together with (2.2.9), we have

$$|y(t,\omega)|^2 \leq |x(t,\omega)|^2 + \int_{t_0}^{t} \frac{(\lambda(s,\omega)y^2(s,\omega))}{[1+(t-s)y^2(s,\omega)]}\,ds. \tag{2.4.30}$$

In the context of the earlier assumption, this is the sharpest inequality one can have. Now, we will show that $|x(t,\omega)|^2$ satisfies the condition (C_6). We note that

$$|x(t,\omega)|^2 = \frac{y_0^2(\omega)}{[1+(t-t_0)y_0^2(\omega)]} \leq \frac{y_0^2(\omega)}{[1+(t-t_0)y_0^2(\omega)]^{1/2}}.$$

By taking the expectation both sides of above inequality and using Schwartz inequality (Lemma A.2.5), we obtain

$$E[|x(t,\omega)|^2] \leq \left(E[|y_0|^2]\right)^{1/2} \left(E\left[\frac{y_0^2(\omega)}{1+(t-t_0)y_0^2(\omega)}\right]\right)^{1/2}. \tag{2.4.31}$$

From the facts that the function $y^2/[1+(t-t_0)y^2]$ is concave and monotonic non-decreasing in y^2 and $y_0 \in L^2[\Omega, R]$, (2.4.31) becomes

$$E[|x(t,\omega)|^2] \leq (E[|y_0(\omega)|^2])^{1/2}\left(\frac{\rho}{[1+(t-t_0)\rho]^{1/2}}\right) \tag{2.4.32}$$

whenever $E[|y_0(\omega)|^2] \leq \rho$, for some $\rho > 0$. It is obvious that $p = 2$, $\alpha(|y_0(\omega)|^2) = \frac{1}{2}\left(|y_0^2(\omega)|\right)^{1/2}$, and $\beta(t-t_0) = (\rho/[1+(t-t_0)\rho])^{1/2}$, $\alpha \in \mathcal{CK}$ and $\beta \in \mathcal{L}$. Because of nonlinearity and dependence of stochastic processes $\lambda(s,\omega)$ and $y(s,\omega)$, (2.4.30) is not tractable to apply Theorem 2.4.3 and its corollaries. However, Theorem 2.4.1 provides an easy and elegant method to attack this problem. Historically, the study of integral inequalities lead to more conservative results than the differential inequalities. This conservatism is reflected in the following discussion. By assuming $\lambda \geq 0$, w.p. 1, we rewrite (2.4.30), as follows

$$|y(t,\omega)|^2 \leq |x(t,\omega)|^2 + \int_{t_0}^{t} \lambda(s,\omega)y^2(s,\omega)ds. \tag{2.4.33}$$

Here, in view of (2.2.9), $\mathcal{D}V(s, v(s,t,y,\omega))$ satisfies

$$|\mathcal{D}V(s, x(t,s,y))| \leq \frac{1}{2}\lambda(s,\omega)y^2.$$

Thus condition (C_3) of Theorem 2.4.2 is valid. Of course, relation (2.4.32) implies (C_4). Therefore, Theorem 2.4.2 is applicable and hence the trivial solution process of (2.2.8) is mean-square stable. This fact in the context of Example 2.4.1 justifies the conservative nature of Theorem 2.4.3 over Theorem 2.4.1.

2.5. ERROR ESTIMATES

First we shall employ the comparison method to derive the error estimates on p-th moment deviation of a solution process of (2.1.1) with the solution process of either (2.1.2) or (2.1.3).

Theorem 2.5.1. *Let the hypotheses of* Theorem 2.2.2 *be satisfied. Further, assume that*

$$b(\|x^p\|) \leq \sum_{i=1}^{m} V_i(t,x,\omega), \tag{2.5.1}$$

where $b \in \mathcal{CK}$ and it is convex function and $p \geq 1$. Then

$$b(\|y(t,\omega)-\bar{x}(t)\|^p) \leq \sum_{i=1}^{m} r_i(t,t_0,V(t_0,x(t,\omega)-\bar{x}(t),\omega),\omega), \qquad t \geq t_0, \tag{2.5.2}$$

and

$$b(E[\|y(t,\omega)-\bar{x}(t)\|^p]) \leq \sum_{i=1}^{m} E[r_i(t,t_0,V(t_0,x(t,\omega)-\bar{x}(t),\omega),\omega)] \quad t \geq t_0. \tag{2.5.3}$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Proof. By the choice of $u_0 = V(t_0, x(t,\omega) - \bar{x}(t), \omega)$, (2.2.34) reduces to

$$\sum_{i=1}^{m} V_i(t, y(t,\omega) - \bar{x}(t), \omega) \leq \sum_{i=1}^{m} r_i(t, t_0, V(t_0, x(t,\omega) - \bar{x}(t), \omega), \omega).$$

This together with (2.5.1) and the convexity of b yields

$$b(E[\|y(t,\omega) - \bar{x}(t)\|^p]) \leq \sum_{i=1}^{m} E[r_i(t, t_0, V(t_0, x(t,\omega) - \bar{x}(t), \omega), \omega)].$$

This completes the proof of theorem.

Example 2.5.1. From Example 2.2.5 and Theorem 2.5.1, we have

$$E[\|y(t,\omega) - \bar{x}(t)\|] \leq E\left[\|x(t,\omega) - \bar{x}(t)\| \exp\left[\int_{t_0}^{t} \frac{1}{2}\gamma(s,\omega)ds\right]\right]$$
$$+E\left[\int_{t_0}^{t} \nu(s,\omega)\|\bar{x}(s)\| \exp\left[\int_{s}^{t} \frac{1}{2}\gamma(u,\omega)du\right] ds\right].$$

This together with Remark A.2.1, yields

$$E[\|y(t,\omega) - \bar{x}(t)\|] \leq$$
$$E\left[\|y_0(\omega) - z_0\| \int_0^1 \exp\left[\int_{t_0}^{t}\left[\mu\Big(\widehat{F}_x(u, x(u, t_0, z_0 + r(y_0(\omega) - z_0)))\Big) + \frac{1}{2}\gamma(u,\omega)\right]du\right]dr\right]$$
$$+E\left[\|z_0\| \int_{t_0}^{t}\int_0^1 \nu(s,\omega) \exp\left[\int_{t_0}^{s} \mu(\widehat{F}_x(u, x(u, t_0, rz_0)))du + \int_{s}^{t} \frac{1}{2}\gamma(u,\omega)du\right] dr\, ds\right]. \quad (2.5.4)$$

Further assume that

$$\int_0^1 \exp\left[\int_{t_0}^{t} \mu(\widehat{F}_y(u, x(u, t_0, z_0 + r(y_0(\omega) - z_0))))du\right] dr \leq$$
$$\leq \exp\left[\int_{t_0}^{t} \Lambda(u,\omega)du\right]. \quad (2.5.5)$$

From (2.5.4) and (2.5.5), we obtain

$$E[\|y(t,\omega)-\bar{x}(t)\|] \leq$$
$$E\left[\|y_0(\omega)-z_0\| \exp\left[\int_{t_0}^{t}(\Lambda(u,\omega)+\frac{1}{2}\nu(u,\omega))du\right]\right]$$
$$+E\left[\|z_0\| \int_{t_0}^{t} \nu(s,\omega)\exp\left[\int_{t_0}^{s}\Lambda(u,\omega)du+\frac{1}{2}\int_{s}^{t}\gamma(u,\omega)du\right]ds\right]. \quad (2.5.6)$$

This together with Hölder inequality, one gets

$$E[\|y(t,\omega)-\bar{x}(t)\|]$$
$$\leq (E[\|y_0(\omega)-z_0\|^2])^{1/2}\left(E\left[\exp\left[\int_{t_0}^{t}(2\Lambda(s,\omega)+\gamma(s,\omega))ds\right]\right]\right)^{1/2}$$
$$+(E[\|z_0\|^2])^{1/2}\left(E\left[\left(\int_{t_0}^{t}\nu(s,\omega)\exp\left[\int_{t_0}^{s}\Lambda(u,\omega)du\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+\int_{s}^{t}\frac{1}{2}\gamma(u,\omega)du\right]ds\right)^2\right]\right)^{1/2}.$$

Further we note that depending on the nature of stochastic processes Λ, γ and ν and random variables $y_0(\omega)$ and z_0, we can obtain specific estimates from (2.5.6). This discussion is applicable to underlining stochastic processes as well as random variables with arbitrary distributions. Moreover, the estimate for absolute mean deviation can be given in terms of the statistical properties of stochastic processes and random variables in (2.5.6). Furthermore, from (2.2.38), one easily obtains the estimate for absolute p^{th} mean deviation with $p \geq 1$. Details are left as an exercise.

To further appreciate this result, we present another example with regard to initial value problems (2.1.18), (2.1.19), and (2.1.20).

Example 2.5.2. From Example 2.2.6 and Theorem 2.5.1, we have

$$E[\|y(t,\omega)-\bar{x}(t)\|] \leq$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

$$E\left[\|x(t,\omega)-\bar{x}(t)\|\exp\left[\int_{t_0}^{t}(\alpha_1(s,\omega)+\beta_1(s,\omega))ds\right]\right]$$
$$+E\left[\int_{t_0}^{t}(\xi_1(s,\omega)+\eta_1(s,\omega))\|\bar{x}(s)\|\right.$$
$$\left.\exp\left[\int_{s}^{t}(\alpha_1(u,\omega)+\beta_1(u,\omega))du\right]ds\right]. \quad (2.5.7)$$

This together with the linearity of the system and the fact

$$\|\Phi(t,s)\| \le \exp\left[\int_s^t \mu(\widehat{A}(u))du\right] \qquad \text{for } t \ge s \ge t_0, \qquad (2.5.8)$$

implies that

$$E[\|y(t,\omega)-\bar{x}(t)\|]$$
$$\le E\left[\|y_0(\omega)-z_0\|\exp\left[\int_{t_0}^{t}(\mu(\widehat{A}(s))+\alpha_1(s,\omega)+\beta_1(s,\omega))ds\right]\right]$$
$$+E\left[\|z_0\|\int_{t_0}^{t}(\xi_1(s,\omega)+\eta_1(s,\omega))\exp\left[\int_{t_0}^{s}\mu(\widehat{A}(u))du\right.\right.$$
$$\left.\left.+\int_{s}^{t}(\alpha_1(u,\omega)+\beta_1(u,\omega))du\right]ds\right]. \quad (2.5.9)$$

From this depending on the distributions of the processes α_1, β_1, ξ_1 and η_1, and random vectors, $y_0(\omega)$ and z_0, one can obtain the estimates for absolute mean deviation of the solution process $y(t,\omega)$ of (2.1.18) with respect to (2.1.19) or (2.1.20). For example, if $\widehat{A}(t) = E[A(t,\omega)]$, then $\beta_1(s,\omega) \equiv \eta_1(s,\omega) \equiv 0$. If we further assume that $y_0(\omega)$, z_0 and $\alpha_1(s,\omega)$ are mutually independent, then (2.5.9) reduces to

$$E\|y(t,\omega)-\bar{x}(t)\| \le$$
$$E[\|y_0(\omega)-z_0\|]E\left[\exp\left[\int_{t_0}^{t}(\mu(\widehat{A}(s))+\alpha_1(s,\omega))ds\right]\right]$$
$$+E[\|z_0\|]E\left[\int_{t_0}^{t}\xi_1(s,\omega)\right.$$

$$\exp\left[\int_{t_0}^{s} \mu(\widehat{A}(u))du + \int_{s}^{t} \alpha_1(u,\omega)du\right] ds\Bigg]. \quad (2.5.10)$$

If we further assume that α_1 is a stationary Gaussian process with mean zero and covariance function $C(t,s)$ and

$$k(t,s) = E\left[\xi_1(s,\omega)\exp\left[\int_{s}^{t} \alpha_1(u,\omega)du\right]\right]$$

exists and it is locally integrable with respect to s. Under these conditions, (2.5.10) reduces to

$$E[\|y(t,\omega) - \bar{x}(t)\|] \leq$$
$$E[\|y_0(\omega) - z_0\|]\exp\left[\int_{t_0}^{t}\left[\mu(\widehat{A}(s)) + \int_{t_0}^{t} C(u,s)du\right] ds\right]$$
$$+E[\|z_0\|]\int_{t_0}^{t} k(t,s)\exp\left[\int_{t_0}^{t} \mu(\widehat{A}(u))du\right] ds. \quad (2.5.11)$$

Problem 2.5.1. For initial value problems (2.1.11), (2.1.13) and (2.1.14), by using Remark 2.2.3 and appropriate conditions show that

$$E\left[\|y(t,\omega) - \bar{x}(t)\|\right] \leq$$
$$E\left[\|y_0(\omega) - z_0\|\exp\Big[\int_{t_0}^{t}\Big[\frac{1}{2}(\alpha(s,\omega) + \beta(s,\omega)) + \gamma(s,\omega)\Big]ds\Big]\right]$$
$$+E\left[\|z_0\|\int_{t_0}^{t}(\zeta(s,\omega) + \eta(s,\omega))\right.$$
$$\left.\exp\left[\int_{t_0}^{s}\gamma(u,\omega)du + \int_{s}^{t}\frac{1}{2}\left[\alpha(u,\omega) + \beta(u,\omega)\right]du\right] ds\right] \quad (2.5.12)$$

for some γ that depends on $\widehat{A}(x,t)$.

We present a few error estimate results by employing the method of variation of parameters.

Theorem 2.5.2. *Suppose that all the hypotheses of* Theorem 2.1.2 *hold. Further, assume that*

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

(i) $b(\|x\|^p) \le \sum_{i=1}^m |V_i(t,x,\omega)| \le a(\|x\|^p)$, (2.5.13)

and

(ii) $\sum_{i=1}^m |[\mathcal{D}V_i(s,x(t,s,y) - x(t,s,z),\omega)]| \le \lambda_1(s,\omega)C(\|y-z\|) + \lambda_2(s,\omega,\|z\|)$ *where* $a \in \mathcal{CK}$ *and it is differentiable,* $b \in \mathcal{VK}$, $C \in \mathcal{K}$ *and* $\mathcal{D}V_i(s,x(t,s,y) - x(t,s,z),\omega)$ *is the i-th component of*

$$\mathcal{D}V(s,x(t,s,y) - x(t,s,z),\omega) = V_s(s,x(t,s,y) - x(t,s,z),\omega) + V_x(s,x(t,s,y) - x(t,s,z),\omega)\Phi(t,s,y)R(t,y,\omega), \quad (2.5.14)$$

$p \ge 1$, $\lambda_1 \in M[R_+, R[\Omega, R_+]]$, $\lambda_2 \in M[R_+ \times R_+, R[\Omega, R_+]]$, *they are sample Lebesgue integrable. Let us define* H, *by* $dH/ds = 1/h(s)$, $h(s) = C((b^{-1}(s))^{1/p})$ *and assume that* $H^{-1} \in \mathcal{CK}$. *Then*

$$\|y(t,\omega) - \bar{x}(t)\|^p \le b^{-1}\left(H^{-1}\left(\int_{t_0}^t \lambda_1(s,\omega)ds + H\left(\gamma(t_0,\omega) + \int_{t_0}^t \beta(s,\omega)ds\right)\right)\right) \quad (2.5.15)$$

and

$$E[\|y(t,\omega) - \bar{x}(t)\|^p] \le J\left(E\left[\int_{t_0}^t \lambda_1(s,\omega)ds\right] + E\left[H\left(a(\|y_0(\omega) - z_0\|^p) + \int_{t_0}^t \beta(s,\omega)ds\right)\right]\right), \quad (2.5.16)$$

for $t \ge t_0$, *where* $J = b^{-1} \circ H^{-1}$, $y(t,\omega)$ *and* $x(t,\omega)$ *are solution process of* (2.1.1) *and* (2.1.3) *through* $(t_0, y_0(\omega))$, *respectively, and* $\bar{x}(t) = x(t,t_0,z_0)$ *is the solution process of either* (2.1.2) *or* (2.1.3) *depending on the choice of* z_0, *that is,* $z_0 = m_0$ *or* $x_0(\omega)$; $\beta(s,\omega)$ *is the absolute value of the sum of* λ_2 *and the time derivative of* $a(\|x(t,\omega) - \bar{x}(t)\|^p)$.

Proof. Let $y(t,\omega) = y(t,t_0,y_0(\omega),\omega)$, $x(t,\omega) = x(t,t_0,y_0(\omega))$, and $\bar{x}(t) = x(t,t_0,z_0)$ be solutions as defined in the theorem, where z_0 is

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

either m_0 or $x_0(\omega)$. From Theorem 2.1.2, (2.5.13) and hypothesis (ii), we obtain

$$b(\|y(t,\omega)-\bar{x}(t)\|^p) \le a(\|x(t,\omega)-\bar{x}(t)\|^p) + \int_{t_0}^{t}\Big(\lambda_1(s,\omega)C(\|y(s,\omega)-\bar{x}(s)\|) + \lambda_2(s,\omega,\|\bar{x}(s)\|)\Big)ds. \quad (2.5.17)$$

Set

$$R(t,\omega) = \int_{t_0}^{t}\Big(C(\|y(s,\omega)-\bar{x}(s)\|)\lambda_1(s,\omega) + \lambda_2(s,\omega,\|\bar{x}(s)\|)\Big)ds.$$

Therefore,

$$R'(t,\omega) = C(\|y(t,\omega)-\bar{x}(t)\|)\lambda_1(t,\omega) + \lambda_2(t,\omega,\|\bar{x}(t)\|),$$

and

$$R(t_0,\omega) = 0. \quad (2.5.18)$$

From (2.5.17), we have

$$b(\|y(t,\omega)-\bar{x}(t)\|^p) \le a(\|x(t,\omega)-\bar{x}(t)\|^p) + R(t,\omega). \quad (2.5.19)$$

This together with (2.5.18), after algebraic computation, yields

$$R'(t,\omega) \le h\Big(a\Big(\|x(t,\omega)-\bar{x}(t)\|^p\Big) + R(t,\omega)\Big)\lambda_1(t,\omega) + \lambda_2(t,\omega,\|\bar{x}(t)\|), \quad (2.5.20)$$

where h is as defined in the theorem. Set

$$u(t,\omega) = a(\|x(t,\omega)-\bar{x}(t)\|^p) + R(t,\omega) \quad (2.5.21)$$

with

$$u(t_0,\omega) = a(\|y_0 - z_0\|^p).$$

Differentiating (2.5.21) on both sides and using (2.5.20), we obtain

$$u'(t,\omega) \le \beta(t,\omega) + h(u(t,\omega))\lambda_1(t,\omega) \quad (2.5.22)$$

with

$$u(t_0, \omega) = a(\|y_0(\omega) - z_0(\omega)\|^p),$$

where $\beta(t, \omega)$ is the absolute value of the sum of $\lambda_2(t, \omega, \|\bar{x}(t)\|)$ and the time derivative of $a(\|x(t, \omega) - \bar{x}(t)\|^p)$. By applying Theorem A.2.4, we have

$$u(t, \omega) \leq r(t, \omega), \qquad t \geq t_0 \text{ w.p. } 1 \tag{2.5.23}$$

where $r(t, \omega)$ is the maximal solution process of

$$\nu' = \beta(t, \omega) + h(\nu)\lambda_1(t, \omega), \qquad \nu(t_0, \omega) = u(t_0, \omega). \tag{2.5.24}$$

Since $h \in \mathcal{K}$ and $\lambda_1 \in M[R_+, R[\Omega, R_+]]$, by an application of Theorem A.2.8, any solution process of (2.5.24) satisfies the following inequality

$$\nu(t, \omega) \leq H^{-1}\left(\int_{t_0}^{t} \lambda_1(s, \omega) ds + H\left(\nu(t_0, \omega) + \int_{t_0}^{t} \beta(s, \omega) ds\right)\right). \tag{2.5.25}$$

From (2.5.19), (2.5.21), (2.5.23) and (2.5.25), we obtain

$$\|y(t, \omega) - \bar{x}(t)\|^p \leq$$
$$b^{-1}\left(H^{-1}\left(\int_{t_0}^{t} \lambda_1(s, \omega) ds + H\left(\nu(t_0, \omega) + \int_{t_0}^{t} \beta(s, \omega) ds\right)\right)\right)$$

which implies that

$$E[\|y(t, \omega) - \bar{x}(t)\|^p] \leq$$
$$J\left(E\left[\int_{t_0}^{t} \lambda_1(s, \omega) ds\right] + E\left[H(a(\|y_0(\omega) - z_0\|^p) + \int_{t_0}^{t} \beta(s, \omega) ds)\right]\right).$$

This completes the proof of the theorem.

Remark 2.5.1. The non-negativity of β is important to obtain estimate (2.5.15). Therefore the definition of β in (2.5.15) deserves a special attention.

Case I: If $z_0 = x_0(\omega)$ and $x_0(\omega) = y_0(\omega)$, then by the uniqueness of solution process of (2.1.3), we conclude that $x(t,\omega) = \bar{x}(t)$. Hence $\beta(s,\omega) \equiv \lambda_2(s,\omega,\|\bar{x}(s)\|)$. Moreover, in this case the estimate (2.5.16) reduces to

$$E[\|y(t,\omega) - x(t,\omega)\|^p] \le J\left(E\left[\int_{t_0}^t \lambda_1(s,\omega)ds\right]\right.$$
$$\left.+E\left[H\left(\int_{t_0}^t \lambda_2(s,\omega,\|\bar{x}(s)\|)ds\right)\right]\right), \qquad \text{for } t \ge t_0. \quad (2.5.26)$$

Case II: In many applications, it is assumed that we have some information about a nominal system (either (2.1.2) or (2.1.3)). Knowing the information we attempt to gather a similar information about the original system, say (2.1.1). In general, the information about the stability of (2.1.2) or (2.1.3) has more importance in engineering applications. In this case, if $a \in \mathcal{CK}$, $V(t,x,\omega) = \|x\|^p$, $p \ge 1$, and in addition either $y_0(\omega) \ne x_0(\omega)$ or $y_0(\omega) \ne m_0$ w.p. 1, then the time derivative of $a(\|x(t,\omega) - \bar{x}(t)\|^p)$ will be non-positive in the context of the stability information of (2.1.3). To justify this, we analyze the crucial part $(x(t,\omega)-\bar{x}(t))^T(\widehat{F}(t,x(t,\omega))-\widehat{F}(t,\bar{x}(t)))$ of the derivative of $a(\|x(t,\omega) - \bar{x}(t)\|^p)$. From hypotheses ($H_2$) of Theorem 2.1.2 and the application Lemmas A.2.1 and A.2.2, we simplify the expression $(x(t,\omega) - \bar{x}(t))^T(\widehat{F}(t,x(t,\omega)) - \widehat{F}(t,\bar{x}(t)))$ as follows:

$$\Big(x(t,\omega) - \bar{x}(t)\Big)^T\Big(\widehat{F}\big(t,x(t,\omega)\big) - \widehat{F}(t,\bar{x}(t))\Big)$$
$$= (x(t,\omega) - \bar{x}(t))^T \widehat{F}_x(t,x(t,\omega),\bar{x}(t))(x(t,\omega) - \bar{x}(t)) \quad (2.5.27)$$

where

$$\widehat{F}_x(t,x(t,\omega),\bar{x}(t)) = \int_0^1 \widehat{F}_x(t,\bar{x}(t) + \delta(x(t,\omega) - \bar{x}(t)))d\delta.$$

From the above mentioned assumption of (2.1.3), it is possible that the quadratic form (2.5.27) is non-positive definite and hence its sum with $\lambda_2(t, \omega, \|\bar{x}(t)\|)$ can be non-positive.

In the context of Remark 2.5.1, Theorem 2.5.2 provides a very conservative estimate. To avoid this conservatism, we present a result which gives a sharper estimate than (2.5.16), and also assures the feasibility of assumption (ii) of Theorem 2.5.2. Moreover, it is in terms of initial data $y_0(\omega)$, $x_0(\omega)$ and m_0 and random rate function.

Corollary 2.5.1. *Suppose that all the hypotheses of* Theorem 2.5.2 *hold except the differentiability of a and assumption* (ii) *are replaced by*

$$\|\Phi(t, s, y)\| \leq K \qquad \text{for } t_0 \leq s \leq t, \;\; y \in R^n, \tag{2.5.28}$$

and assumption (ii) *is valid whenever* (2.5.28) *holds, where* K *is a positive real number. Then* (2.5.16) *reduces to*

$$E[\|y(t, \omega) - \bar{x}(t)\|^p] \leq J\Bigg(E\left[\int_{t_0}^{t} \lambda_1(s, \omega) ds\right]$$
$$+E\left[H\left(a(K^p\|y_0(\omega) - z_0\|^p) + \int_{t_0}^{t} \lambda_2(s, \omega, \|\bar{x}(s)\|) ds\right)\right]\Bigg) \tag{2.5.29}$$

for $t \geq t_0$.

Proof. From Lemma A.2.2, we have

$$x(t, t_0, y_0(\omega)) - x(t, t_0, z_0) =$$
$$\int_0^1 \Phi(t, t_0, y_0(\omega) + s(y_0(\omega) - z_0)) ds (y_0(\omega) - z_0).$$

This together with (2.5.28) yields

$$\|x(t, \omega) - \bar{x}(t)\| \leq K\|y_0(\omega) - z_0\|,$$

and hence

$$a(\|x(t,\omega)-\bar{x}(t)\|^p) \le a(K^p\|y_0(\omega)-z_0\|^p). \qquad (2.5.30)$$

From (2.5.17) and (2.5.30), we have

$$b(\|y(t,\omega)-\bar{x}(t)\|^p) \le a\Big(K^p\|y_0(\omega)-z_0\|^p\Big) \\ + \int_{t_0}^{t}\Big(\lambda_1(s,\omega)C(\|y(s,\omega)-\bar{x}(s)\|^p)+\lambda_2(s,\omega,\|\bar{x}(s)\|)\Big)ds.$$

By setting

$$u(t,\omega)=a(K^p\|y_0(\omega)-z_0\|^p)+R(t,\omega), \quad u(t_0,\omega)=a(K^p\|y_0(\omega)-z\|^p),$$

$\beta(t,\omega)=\lambda_2(t,\omega,\|\bar{x}(t)\|)$ w.p. 1. Then by application of Theorem 2.5.2, we establish inequality (2.5.29).

In the next theorem we modify hypotheses (ii) of Theorem 2.5.2.

Theorem 2.5.3. *Assume that all the hypotheses of* Theorem 2.5.2 *hold except hypothesis* (ii) *is replaced by*

$$\sum_{i=1}^{m}|\mathcal{D}V_i(s,x(t,s,y)-x(t,s,z),\omega)| \le \\ \lambda_1(s,\omega)C\left(\sum_{i=1}^{m}|V_i(s,y-z,\omega)|\right)+\lambda_2(s,\omega,\|z\|), \qquad (2.5.31)$$

where $\mathcal{D}V_i(s,x(t,s,y)-x(t,s,z),\omega)$, C and λ_i are as defined before. Define $H(s)=\int\frac{ds}{C(s)}$. Then the conclusion of Theorem 2.5.2 *remains valid.*

Proof. By following the proof of Theorem 2.1.2 and using (2.5.31), we arrive at

$$\sum_{i=1}^{m}|V_i(t,y(t,\omega)-\bar{x}(t),\omega)| \le \sum_{i=1}^{m}|V_i(t_0,x(t,\omega)-\bar{x}(t),\omega)|$$

$$+\int_{t_0}^{t}\left(C\left(\sum_{i=1}^{m}V_i(s,y(s,\omega)-\bar{x}(s),\omega)\right)\right.$$

$$\left.\lambda_1(s,\omega)+\lambda_2(s,\omega,\|\bar{x}(s)\|)\right)ds. \quad (2.5.32)$$

Set

$R(t,\omega) =$

$$\int_{t_0}^{t}\left(C\left(\sum_{i=1}^{m}V_i(s,y(s,\omega)-\bar{x}(s),\omega)\right)\lambda_1(s,\omega)+\lambda_2(s,\omega,\|\bar{x}(s)\|)\right)ds.$$

From (2.5.32) and hypothesis (i), we obtain

$$\sum_{i=1}^{m}|V_i(t,y(t,\omega)-\bar{x}(t),\omega)| \le a(\|x(t,\omega)-\bar{x}(t)\|^p)+R(t,\omega).$$

Now the proof of the theorem follows by applying the method used in Theorem 2.5.2.

Corollary 2.5.2. *Assume that the hypotheses of* Theorem 2.5.3 *are satisfied except the differentiability of a is replaced by relation* (2.5.28), *and relation* (2.5.31) *holds whenever* (2.5.28) *is valid. Then* (2.5.29) *remains true.*

The proof of the corollary is left as an exercise.

To illustrate the feasibility of assumption (ii) of Theorem 2.5.2 and its variation in Theorem 2.5.3, we state a remark.

Remark 2.5.2. First we remark that assumption (ii) and its variations are feasible provided that solution processes of (2.1.1) are bounded w.p. 1. Further, we note that if the trivial solution process of (2.1.1) is stable in probability one then these conditions are also feasible. In other words, assumption (ii) and its variant are verifiable whenever they satisfy for $\|y\| < \rho$ some $\rho > 0$. In addition, by using the Jensen's inequality, the results concerning stability in probability can be derived.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

To justify Remark 2.5.2 and the fruitfulness of above results, we present a few examples.

Example 2.5.3. Let $V(t,x) = \|x\|^2$. In this case, $b(r) = a(r) = r$, $p = 2$, and $\mathcal{D}V(s, x(t,s,y) - \bar{x}(t))$ relative to (2.1.1) is given by

$$\mathcal{D}V(s, x(t,s,y) - \bar{x}(t)) = 2(x(t,s,y) - \bar{x}(t))^T \Phi(t,s,y) R(s,y,\omega)$$

and assume that

$$|\mathcal{D}V(s, x(t,s,y) - x(t,s,z))| \le \|x(t,s,y) - x(t,s,z)\| \lambda(s,\omega) \quad (2.5.33)$$

for $t_0 \le s \le t$ and $\|y\| \le \rho$ for some $\rho > 0$. Here $H(r) = C(r) = r^{1/2}$ and $\lambda \in M[R_+, R[\Omega, R_+]]$ and it is sample Lebesgue integrable. Further assume that $\Phi(t,s,y)$ satisfies relation (2.5.28), and moreover the solution process $y(t,\omega)$ of (2.1.1) is bounded with probability one. From (2.5.28), Lemma A.2.2, (2.5.33) reduces to

$$|\mathcal{D}V(s, x(t,s,y) - x(t,s,z))| \le K\|y - z\| \lambda(s,\omega),$$
$$\text{for } t_0 \le s \le t \text{ and } \|y\| \le \rho.$$

Now by an application of Corollary 2.5.1, we get

$$\|y(t,\omega) - \bar{x}(t)\|^2 \le \left[K \int_{t_0}^{t} \lambda(s,\omega) ds + K\|y_0(\omega) - z_0\| \right]^2, \quad \text{for } t \ge t_0. \quad (2.5.34)$$

Taking expectation on both sides of (2.5.34), we have

$$E[\|y(t,\omega) - \bar{x}(t)\|^2] \le E\left[K \left(\int_{t_0}^{t} \lambda(s,\omega) ds + \|y_0(\omega) - z_0\| \right) \right]^2$$
$$\text{for } t \ge t_0. \quad (2.5.35)$$

Remark 2.5.3. We remark that by selecting various conditions on the process $\lambda(s,\omega)$ and random variable $\|y_0(\omega) - z_0\|$ one can obtain more attractive estimates. For example:

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

(i) Taking the square root and expectation on both sides of (2.5.34), we obtain

$$E[\|y(t,\omega)-\bar{x}(t)\|] \le K\left(E[\|y_0(\omega)-z_0\|]+E\left[\int_{t_0}^{t}\lambda(s,\omega)ds\right]\right) \tag{2.5.36}$$

for $t \ge t_0$.

(ii) When $\|y_0(\omega)-z_0\|$ and $\lambda(s,\omega)$ are second order independent random processes, from (2.5.35), we obtain

$$E[\|y(t,\omega)-\bar{x}(t)\|^2] \le K^2\Bigg(E[\|y_0(\omega)-z_0\|^2]$$
$$+2E[\|y_0(\omega)-z_0\|]E\left[\int_{t_0}^{t}\lambda(s,\omega)ds\right]+E\left[\left(\int_{t_0}^{t}\lambda(s,\omega)ds\right)^2\right]\Bigg) \tag{2.5.37}$$

for $t \ge t_0$. Further, if $\lambda(s,\omega)$ is a zero mean stationary Gaussian process then $E[\lambda(s,\omega)\lambda(u,\omega)]$ depends only on $s-u$. Hence (2.5.37) reduces to

$$E[\|y(t,\omega)-\bar{x}(t)\|^2] \le$$
$$K^2\left(E[\|y_0(\omega)-z_0\|^2]+\int_{t_0}^{t}\int_{t_0}^{t}C(u-s)du\,ds\right) \tag{2.5.38}$$

for $t \ge t_0$, where $E[\lambda(u,\omega)\lambda(s,\omega)] = C(u-s)$.

Example 2.5.4. Suppose $V(t,x) = \|x\|^p$, for $p \ge 1$. Assume that the solution process (2.1.1) is bounded w.p. 1, and $\Phi(t,s,y)$ satisfies relation (2.5.28). Here

$$\mathcal{D}V(s,x(t,s,y)-x(t,s,z))$$
$$= p\|x(t,s,y)-x(t,s,z)\|^{p-2}(x(t,s,y)-x(t,s,z))^T\Phi(t,s,y)R(s,y,\omega)$$

and assume that

$$|\mathcal{D}V(s,x(t,s,y)-x(t,s,z))| \le C(V(t,y-z))\lambda(s,\omega)$$

provided (2.5.28) is satisfied and $\|y\| \leq \rho$ for some $\rho > 0$, where $C(r) = pK^p(r)^{(p-1)/p}$. Again, $a(s) = b(s) = s$, hence $h(s) = C(s)$ and λ is defined as before. By applying Theorem 2.5.3 in context of Corollary 2.5.1, we obtain

$$E[\|y(t,\omega) - \bar{x}(t)\|^p] \leq \\ E\left[K\left(\int_{t_0}^{t} \lambda(s,\omega)ds + \|y_0(\omega) - z_0\|\right)\right]^p, \quad t \geq t_0. \quad (2.5.39)$$

Using the known inequality $(\alpha+\beta)^p \leq 2^p(\alpha^p + \beta^p)$, (2.5.39) reduces to

$$E[\|y(t,\omega) - \bar{x}(t)\|^p] \leq \\ (2K)^p \left(E\left[\int_{t_0}^{t} \lambda(s,\omega)\right]^p + E[\|y_0(\omega) - z_0\|^p],\right) \quad t \geq t_0. \quad (2.5.40)$$

Thus far we have presented examples in the context of (2.1.1) and (2.1.3) with a different class of nonlinearities. In the following, we give additional simple illustrative examples.

Example 2.5.5. Consider the following differential equation with random coefficients

$$y'(t,\omega) = A(t,\omega)y(t,\omega), \qquad y(t_0,\omega) = y_0(\omega) \qquad (2.5.41)$$

where $A \in M[R, R[\Omega, R]]$ and satisfies enough regularity conditions for existence of sample solution process of (2.5.41).

$$x' = E[A(t,\omega)]x, \qquad x(t_0) = z_0, \qquad (2.5.42)$$

where z_0 is either $E[y_0(\omega)]$ or $x_0(\omega) \neq y_0(\omega)$ w.p. 1. Here

$$x(t,\omega) = y_0(\omega)\exp\left[\int_{t_0}^{t} E[A(s,\omega)]ds\right],$$

$$\Phi(t,s,y) = \exp\left[\int_s^t E[A(u,\omega)]du\right].$$

Suppose $V(t,x) = |x|$ and hence

$$V(s, x(t,s,y) - x(t,s,z)) = \exp\left[\int_s^t E[A(u,\omega)]du\right]|y-z|.$$

Then

$$\mathcal{D}V(s, x(t,s,y) - x(t,s,z)) = \text{sign}(y-z)\Phi(t,s,y)R(s,y,\omega)$$

where $R(s,y,\omega) = (A(s,\omega) - E[A(s,\omega)])y$. It is obvious that after certain algebraic computations, $\mathcal{D}V(s,x(t,s,y) - x(t,s,z))$ satisfies the following relation

$$|\mathcal{D}V(s,x(t,s,y) - x(t,s,z))| \leq \lambda_1(s,\omega)|y - \bar{x}(s)| + \lambda_2(s,\omega,|\bar{x}(s)|)$$

where $\lambda_1(s,\omega) = |A(s,\omega) - E[A(s,\omega)]|$ and

$$\lambda_2(s,\omega,|\bar{x}(s)|) = |A(s,\omega) - E[A(s,\omega)]||\bar{x}(s)|.$$

Assume that $\exp\left[\int_{t_0}^t E[A(u,\omega)]du\right] \leq K$. In the light of these considerations, we apply Corollary 2.5.1 and conclude that

$$E[|y(t,\omega) - \bar{x}(t)|] \leq$$
$$E\left[\left(K|y_0(\omega) - z_0| + \int_{t_0}^t \lambda_2(s,\omega,|\bar{x}(s)|)ds\right)\exp\left[\int_{t_0}^t \lambda_1(s,\omega)ds\right]\right].$$

Here $C(s) = s = a(s) = b(s)$, $\frac{dH}{ds} = \frac{1}{s}$.

Example 2.5.6. Consider the following differential equation

$$y'(t,\omega) = -\frac{1}{2}A(t,\omega)y^3(t,\omega), \qquad y(t_0,\omega) = y_0(\omega) \tag{2.5.43}$$

where $A \in M[R, R[\Omega, R_+]]$ and satisfies enough regularity conditions for existence of sample solution process for $t \geq t_0$. Consider the smooth differential equation corresponding to (2.5.43),

$$x' = -\frac{1}{2}E[A(t,\omega)]x^3, \qquad x(t_0) = z_0. \tag{2.5.44}$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

We note that $x(t,\omega)$, $y(t,\omega)$, and $\Phi(t,s,y)$ are given by

$$x(t,\omega) = \frac{y_0(\omega)}{\left[1 + y_0^2(\omega)\int_{t_0}^t E[A(s,\omega)]ds\right]^{1/2}},$$

$$\Phi(t,s,y) = \frac{1}{\left[1 + y^2\int_s^t E[A(u,\omega)]du\right]^{3/2}},$$

and

$$y(s,\omega) = \frac{y_0(\omega)}{\left[1 + y_0^2(\omega)\int_{t_0}^s A(s,\omega)ds\right]^{1/2}}, \qquad t \geq s \geq t_0.$$

It is obvious that $y(s,\omega)$ is bounded with probability one if $y_0(\omega)$ is bounded w.p. 1. For $V(t,x) = |x|$, $a(s) = b(s) = s$, $\frac{dH(s)}{ds} = 1$, $|\Phi(t,s,y)| \leq 1$, and as in Example 2.5.5

$$\begin{aligned} |\mathcal{D}V(s, x(t,s,y) - x(t,s,z))| &\leq |\Phi(t,s,y)R(s,y,\omega)| \\ &\leq K|A(s,\omega) - E[A(s,\omega)]| \end{aligned}$$

provided $|y_0(\omega)| \leq \rho$, where

$$R(s,y,\omega) = \frac{1}{2}(A(s,\omega) - E[A(s,\omega)])y^3.$$

By applying a simplified version of Corollary 2.5.1, we get

$$|y(t,\omega) - \bar{x}(t)| \leq |y_0(\omega) - z_0| + K\left[\int_{t_0}^t (A(s,\omega) - E[A(s,\omega)])ds\right] \tag{2.5.45}$$

for $t \geq t_0$, which implies

$$E[|y(t,\omega) - \bar{x}(t)|] \leq E[|y_0(\omega) - z_0|] + KE\left[\int_{t_0}^t (A(s,\omega) - E[A(s,\omega)])ds\right] \tag{2.5.46}$$

for $t \geq t_0$.

In particular, if random processes $|y_0(\omega) - z_0|$ and $|A(s,\omega) - E[A(s,\omega)]|$ follow certain additional conditions, we get the following interesting inequalities:

(i) If $|y_0(\omega) - z_0|$ and $|A(s,\omega) - E[A(s,\omega)]|$ are independent processes, then from (2.5.45) and using $(\alpha+\beta)^2 \leq 2(\alpha^2+\beta^2)$, we obtain

$$E[|y(t,\omega) - \bar{x}(t)|^2] \leq 2\left[E\left[|y_0(\omega) - z_0|^2\right] + K^2 E\left[\left(\int_{t_0}^{t}(A(s,\omega) - E[A(s,\omega)])ds\right)^2\right]\right] \tag{2.5.47}$$

for $t \geq t_0$.

(ii) If $|A(s,\omega) - E[A(s,\omega)]|$ and $|y_0(\omega) - z_0|$ are uncorrelated and$|A(s,\omega) - E[A(s,\omega)]|$ is a stationary Gaussian process, then from (2.5.47) we get

$$E[|y(t,\omega) - \bar{x}(t)|^2] \leq 2\left[E[|y_0(\omega) - z_0|^2] + K^2 \int_{t_0}^{t}\int_{t_0}^{t} C(s-u)du\,ds\right], \tag{2.5.48}$$

for $t \geq t_0$, where $C(s-u) = E[|A(s,\omega) - E[A(s,\omega)]||A(u,\omega) - E[A(u,\omega)]|]$.

Remark 2.5.4. We note that our results also provide an error estimate in the sense of probability one and in probability. This statement can be justified by inequalities (2.5.2), (2.5.15), (2.5.34) and (2.5.45).

2.6. RELATIVE STABILITY

We shall present a few relative stability criteria that assures the relative stability of the p^{th} moment of systems of differential equations (2.1.1) with respect to either system (2.1.2) or (2.1.3). These results also provide sharper error estimates on p^{th} moment deviation of a

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

solution process of (2.1.1) relative to either system (2.1.2) or (2.1.3). Furthermore, examples are given to illustrate the results.

In the following, we present relative stability results in the context of comparison method.

Theorem 2.6.1. *Let the hypotheses of* Theorem 2.2.2 *be satisfied. Further assume that for* $(t, x) \in R_+ \times R^n$

$$b(\|x\|^p) \leq \sum_{i=1}^{m} V_i(t, x, \omega) \leq a(t, \|x\|^p), \tag{2.6.1}$$

where $b \in \mathcal{VK}$, $a \in \mathcal{CK}$, *and* $p \geq 1$. *Then*

(i) (JR_1) *of* (2.2.2) *and* (2.1.2) *implies* (RM_1) *of* (2.1.1) *and* (2.1.2);

(ii) (JR_2) *of* (2.2.2) *and* (2.1.2) *implies* (RM_2) *of* (2.1.1) *and* (2.1.2).

Proof. Let $\epsilon > 0$, $t_0 \in R_+$ be given. From the joint relative stability of (2.2.2) and (2.1.2), for given $b(\epsilon^p) > 0$, $t_0 \in R_+$, one can find $\delta_1 = \delta_1(\epsilon, t_0) > 0$ such that

$$\sum_{i=1}^{m} E\Big[\nu_i(t_0, t_0, y_0 - z_0, \omega)\Big] \leq \delta_1 \tag{2.6.2}$$

implies

$$\sum_{i=1}^{m} E[\nu_i(t, t_0, y_0(\omega) - z_0, \omega)] < b(\epsilon^p), \qquad t \geq t_0 \tag{2.6.3}$$

whenever $\|z_0\|$ is small enough, where

$$\nu(t, t_0, y_0(\omega) - z_0, \omega) = u(t, t_0, V(t_0, x(t, \omega) - \bar{x}(t), \omega), \omega)$$

and $u(t, t_0, u_0(\omega), \omega)$ is a solution process of (2.2.2). For $\delta_1(\epsilon)$, we can find $\delta(\epsilon) > 0$ such that $a(t_0, E[\|y_0 - z_0\|^p]) < \delta_1(\epsilon)$, whenever $\|y_0 - z_0\|_p < \delta$. Now we claim that if $\|y_0 - z_0\|_p < \delta$ is valid then

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

$\|y(t,\omega) - \bar{x}(t)\|_p < \epsilon$ for all $t \geq t_0$. Suppose that this is false. Then there would exist solution processes $y(t,\omega)$ and $\bar{x}(t)$ satisfying (2.6.2) such that

$$\|y(t_1,\omega) - \bar{x}(t_1)\|_p = \epsilon \qquad \text{for some } t_1 > t_0, \tag{2.6.4}$$

and

$$\|y(t,\omega) - \bar{x}(t)\|_p \leq \epsilon \qquad \text{for } t \in [t_0, t_1]. \tag{2.6.5}$$

From the conclusion of Theorem 2.2.2 and relations (2.6.1), (2.6.4), we have

$$\begin{aligned} b(\|y(t_1,\omega) - \bar{x}(t_1)\|^p) &\leq \sum_{i=1}^{m} E[V_i(t_1, y(t_1,\omega) - \bar{x}(t_1), \omega)] \\ &\leq \sum_{i=1}^{m} E[r_i(t_1, t_0, u_0(\omega), \omega)]. \end{aligned}$$

This together with (2.6.3) and (2.6.4) leads to the contradiction

$$\begin{aligned} b(\epsilon^p) &\leq \sum_{i=1}^{m} E[V_i(t_1, y(t_1,\omega) - \bar{x}(t_1), \omega)] \\ &\leq \sum_{i=1}^{m} E[r_i(t_1, t_0, u_0(\omega), \omega)] < b(\epsilon^p). \end{aligned}$$

This completes the proof of (i). The proof of (ii) can be reformulated, analogously. The proof of the theorem is complete.

Example 2.6.1. We consider Examples 2.2.5 and 2.5.1. In this case $p = 1$, $a(r) = b(r) = r$, and the comparison equation is as described in (2.2.37). To apply Theorem 2.6.1, it is enough to show that (2.2.37) and (2.1.13) (or (2.1.14)) are jointly relatively stable in the mean. For this purpose, we assume

$$E\left[\exp\left[\int_{t_0}^{t} (\Lambda(s,\omega) + \frac{1}{2}\,\nu(s,\omega))ds\right]\right]$$

and

$$E\left[\int_{t_0}^{t} \nu(s,\omega)\exp\left[\int_{t_0}^{s} \Lambda(u,\omega)du + \int_{s}^{t} \frac{1}{2}\,\nu(u,\omega)du\right] ds\right]$$

are bounded on $[t_0, +\infty)$. This together with the right-hand side expression in (2.5.5) and Hölder inequality establishes the (JR$_1$) property of (2.2.37) and (2.1.13) (or (2.1.14)). Thus the (RM$_1$) property of (2.1.11) and (2.1.13) (or (2.1.14)) follows from the application of Theorem 2.6.1.

Example 2.6.2. Consider the initial value problems (2.1.18), (2.1.19) and (2.1.20). From Examples 2.2.6 and 2.5.2, we can arrive at (2.5.9). To conclude the equations (2.2.47) and (2.1.19) (or 2.1.20)) have (JR$_1$) property, we assume that the moments of

$$\exp\left[\int_{t_0}^{t} \alpha_1(s,\omega)ds\right]$$

and

$$\int_{t_0}^{t} \xi_1(s,\omega)\exp\left[\int_{t_0}^{s} \mu(\widehat{A}(u))du + \int_{s}^{t} \alpha_1(u,\omega)du\right]$$

are bounded on $[t_0, +\infty)$. Again, by the application of Theorem 2.6.1 we conclude that (2.1.18) and (2.1.19) (or (2.1.20)) have (RM$_1$) property.

Example 2.6.3. We consider again the initial value problems (2.1.18), (2.1.19) and (2.1.20). We assume that $\widehat{A}(s) = E[A(s,\omega)]$, and

$$\|\Phi(t,s)(A(s,\omega) - \widehat{A}(s))\Phi(s,t)\| \le \lambda(s,\omega), \qquad \text{for } t \ge s \ge t_0 \quad (2.6.6)$$

where $\lambda \in M[R_+, R[\Omega, R]]$, and it is independent of $y_0(\omega)$ and z_0. Furthermore it satisfies

$$E\left[\exp\left[\int_{t_0}^{t} \lambda(s,\omega)ds\right]\right] \le K\exp\left[\int_{t_0}^{t} \widehat{\lambda}(s)ds\right], \qquad (2.6.7)$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

and

$$\lim_{t\to\infty}\left[\frac{1}{t-t_0}\int_{t_0}^{t}(\widehat{\lambda}(s)+\mu(\widehat{A}(s)))ds\right]\le -\alpha \tag{2.6.8}$$

for some positive real number α. Let $V(t,x,\omega)=\|x\|$. By following the argument of Example 2.2.6, we obtain

$$D^{+}V(s,x(t,s,y)-x(t,s,z),\omega)\le \lambda(s,\omega)[V(s,x(t,s,y)-x(t,s,z),\omega)+V(s,x(t,s,z),\omega)].$$

The comparison equation is

$$u'=\lambda(s,\omega)u+\lambda(s,\omega)V(s,x(t,s,z),\omega),\qquad u(t_0)=u_0. \tag{2.6.9}$$

The solution process of this equation is

$$u(s,t_0,u_0,\omega)=u_0\exp\left[\int_{t_0}^{s}\lambda(v,\omega)dv\right] + \int_{t_0}^{s}\lambda(v,\omega)V(v,x(t,v,\bar{x}(v)),\omega)\exp\left[\int_{v}^{s}\lambda(w,\omega)dw\right]dv. \tag{2.6.10}$$

To conclude the (JR_2) of comparison equation (2.6.9) with (2.1.19) (or (2.1.20)), we assume that y_0 and A are independent processes and we observe

$$\begin{aligned}V(v,x(t,v,\bar{x}(v)),\omega)&=\|x(t,v,\bar{x}(v))\|=\|x(t,t_0,z_0)\|\\&\le K\|z_0\|\exp\left[\int_{t_0}^{t}\mu(\widehat{A}(s))ds\right].\end{aligned}$$

This together with sample continuity of $u(s,u_0)$, Remark 2.2.1, (2.6.7) and (2.6.10) yields

$$\begin{aligned}&E[u(t,t_0,u_0,\omega)]\\&\quad\le E\left[\|x(t,\omega)-\bar{x}(t)\|+\|x(t,t_0,z_0)\|\exp\left[\int_{t_0}^{t}\lambda(s,\omega)ds\right]\right]\\&\quad\le E[\|y_0(\omega)-z_0\|+\|z_0\|]K\exp\left[\int_{t_0}^{t}(\widehat{\lambda}(s)+\mu(\widehat{A}(s)))ds\right].\end{aligned}$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

From this, (2.6.8) and following standard argument one can conclude that (2.6.9) and (2.1.19) (or (2.1.20)) are jointly relatively asymptotically stable in the mean. Therefore by the application of Theorem 2.6.1, we conclude that systems (2.1.18) and (2.1.19) (or (2.1.20)) are relatively asymptotically stable in the mean square.

Now we employ the variation of constants method to investigate the p^{th} moment relative stability of (2.1.1) with respect to either system (2.1.2) or (2.1.3).

Theorem 2.6.2. *Suppose that all the hypotheses of* Theorem 2.1.2 *hold. Moreover, assume that*

(i) $\widehat{F}(t, 0) \equiv 0$ *for* $t \in R_+$;

(ii) $b(\|x\|^p) \leq \sum_{i=1}^m |V_i(t, x, \omega)| \leq a(\|x\|^p)$ *for* $(t, x) \in R_+ \times R^n$, *where* $b \in \mathcal{VK}$, $a \in \mathcal{K}$ *and* $p \geq 1$;

(iii) $\sum_{i=1}^m |\mathcal{D}V_i(s, x(t, s, y) - x(t, s, z), \omega)| \leq$ $\eta(t-s)\Big[\lambda_1(s, \omega) \left(\sum_{i=1}^m |V_i(s, y - z, \omega)|\right)$ $+\lambda_2(s, \omega) \left(\sum_{i=1}^m |V_i(s, z, \omega)|\right)\Big]$ *for* $t_0 \leq s \leq t$, $\|y - z\|^p < \rho$, $\|z\|^p < \rho$ *some* $\rho > 0$, $\lambda_i \in M[R_+, R[\Omega, R_+]]$ *and* λ_i *are sample integrable,* $i = 1, 2$, *and* $\eta \in \mathcal{L}$;

(iv) $\sum_{i=1}^m |V_i(s, x(t, \omega) - \bar{x}(t), \omega)| \leq \alpha(\|y_0(\omega) - z_0\|^p)\beta(t - t_0)$, $t \geq s \geq t_0$, *whenever* $E[\|y_0(\omega) - z_0\|^p] < \rho$ *and* $E[\|z_0\|^p] < \rho$, *where* $\beta \in \mathcal{L}$ *and* $\alpha \in \mathcal{CK}$;

(v) *there exists a positive number* K *such that*

$$\eta(t - s)\beta(s - t_0) \leq K\beta(t - t_0)$$

where η *and* β *are as defined above;*

(vi) *the processes* λ_i *are independent of* y_0 *and* z_0. *Then*

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

(1) *the boundedness of* $\beta(t-t_0)E\left[\exp\left[K\int_{t_0}^t \lambda_1(s,\omega)ds\right]\right]$ *and*

$$\beta(t-t_0)E\left[\int_{t_0}^t \lambda_2(s,\omega)\exp\left[\int_s^t \lambda_1(u,\omega)du\right]ds\right]$$

imply (RM_1) *of* (2.1.1) *and* (2.1.2);

(2) $\lim_{t\to\infty}\left[\beta(t-t_0)E\left[\exp\left[K\int_{t_0}^t \lambda_1(s,\omega)\right]ds\right]\right]=0$ *and*

$$\lim_{t\to\infty}\left[\beta(t-t_0)E\left[\int_{t_0}^t \lambda_2(s,\omega)\exp\left[K\int_s^t \lambda_1(u,\omega)du\right]ds\right]\right]=0$$

imply (RM_2) *of* (2.1.1) *and* (2.1.2).

Proof. Let $y(t,\omega)=y(t,t_0,y_0(\omega),\omega)$, $x(t,\omega)=x(t,t_0,y_0(\omega))$ and $\bar{x}(t)=x(t,t_0,z_0)$ be the solution processes as defined in Theorem 2.1.2. From Theorem 2.1.2 and hypothesis (iii), we have

$$\begin{aligned}\sum_{i=1}^m |V_i(t,y(t,\omega)-\bar{x}(t),\omega)| &\le \sum_{i=1}^m |V_i(t_0,x(t,\omega)-\bar{x}(t),\omega)| \\ &\quad+\int_{t_0}^t \eta(t-s)\Bigg[\lambda_1(s,\omega)\left(\sum_{i=1}^m |V_i(s,y(s,\omega)-\bar{x}(s),\omega)|\right) \\ &\quad+\lambda_2(s,\omega)\left(\sum_{i=1}^m |V_i(s,\bar{x}(s),\omega)|\right)\Bigg]ds. \qquad (2.6.11)\end{aligned}$$

as long as $E[\|y(t,\omega)\|^p]\le\rho$ and $E[\|\bar{x}(t)\|^p]\le\rho$. From hypotheses (i) and (iv), we note that

$$\sum_{i=1}^m |V_i(s,\bar{x}(s),\omega)| \le \alpha(\|z_0\|^p)\beta(s-t_0), \qquad t\ge t_0. \qquad (2.6.12)$$

By setting $m(t,\omega)=\frac{1}{\beta(t-t_0)}\left(\sum_{i=1}^m |V_i(t,y(t,\omega)-\bar{x}(t),\omega)|\right)$ and using hypotheses (iv), (v) and relation (2.6.12), relation (2.6.11) reduces to

$$\begin{aligned}m(t,\omega) &\le \alpha(\|y_0(\omega)-z_0\|^p) \\ &\quad+K\int_{t_0}^t [\lambda_1(s,\omega)m(s,\omega)+\lambda_2(s,\omega)\alpha(\|z_0\|^p)]\,ds\end{aligned} \qquad (2.6.13)$$

as long as $E[\|y(t,\omega)\|^p] \le \rho$ and $E[\|\bar{x}(t)\|^p] \le \rho$. Set

$$R(t,\omega) = \alpha(\|y_0(\omega) - z_0\|^p) + K\int_{t_0}^{t} [\lambda_1(s,\omega)m(s,\omega) + \lambda_2(s,\omega)\alpha(\|z_0\|^p)]\,ds.$$

Hence

$$R'(t,\omega) = K[\lambda_1(t,\omega)m(t,\omega) + \lambda_2(t,\omega)\alpha(\|z_0\|^p)]$$

and

$$R(t_0,\omega) = \alpha(\|y_0(\omega) - z_0\|^p).$$

This together with (2.6.13), one obtains

$$R'(t,\omega) \le K\lambda_1(t,\omega)R(t,\omega) + K\lambda_2(t,\omega)\alpha(\|z_0\|^p)$$

with

$$R(t_0,\omega) = \alpha(\|y_0(\omega) - z_0\|^p).$$

By solving this random differential inequality, we obtain

$$R(t,\omega) \le \alpha(\|y_0(\omega) - z_0\|^p)\exp\left[K\int_{t_0}^{t}\lambda_1(s,\omega)ds\right] + K\alpha(\|z_0\|^p)\int_{t_0}^{t}\lambda_2(s,\omega)\exp\left[K\int_{s}^{t}\lambda_1(u,\omega)du\right]ds.$$

This together with the definition of $m(t,\omega)$, (2.6.13) reduces to

$$\sum_{i=1}^{m}|V_i(t,y(t,\omega) - \bar{x}(t),\omega)| \le \beta(t-t_0)\left(\alpha(\|y_0(\omega) - z_0\|^p)\exp\left[K\int_{t_0}^{t}\lambda_1(s,\omega)ds\right] + K\alpha(\|z_0\|^p)\int_{t_0}^{t}\lambda_2(s,\omega)\exp\left[K\int_{s}^{t}\lambda_1(u,\omega)du\right]ds\right). \quad (2.6.14)$$

From hypotheses (ii), (vi), and (2.6.14), we get

$$b(E[\|y(t,\omega)-\bar{x}(t)\|^p]) \leq \beta(t-t_0)\Bigg(\alpha(E[\|y_0(\omega)-z_0\|^p])E\left[\exp\left[K\int_{t_0}^{t}\lambda_1(s,\omega)ds\right]\right] + K\alpha(E[\|z_0\|^p])E\left[\int_{t_0}^{t}\lambda_2(s,\omega)\exp\left[\int_{s}^{t}\lambda_1(u,\omega)du\right]ds\right]\Bigg) \quad (2.6.15)$$

as long as $E[\|y(t,\omega)\|^p] \leq \rho$ and $E[\|\bar{x}(t)\|^p] \leq \rho$. It is obvious that ($RM_1$) and ($RM_2$) properties of (2.1.1) and (2.1.2) (or (2.1.3)) follows from the given assumptions on β and λ_i for $i = 1, 2$ and by following the argument used in the proof of Theorem 2.4.3. This completes the proof of the theorem.

Remark 2.6.1. We remark that Theorems 2.6.1 and 2.6.2 also provide the sufficient conditions for relative stability properties of (2.1.1) and (2.1.2) (or (2.1.3)) with probability one as well as in probability.

Remark 2.6.2. The relative stability conditions about β and λ_i can be further simplified by considering $\lambda = \max\{\lambda_1, \lambda_2\}$. With this modification, the relative stability conditions in (1) and (2) in Theorem 2.6.2 reduces to

(1*) the boundedness of

$$\beta(t-t_0)E\left[\exp\left[K\int_{t_0}^{t}\lambda(s,\omega)ds\right]\right]$$

and

(2*) $\lim_{t\to\infty}\beta(t-t_0)E\left[\exp\left[K\int_{t_0}^{t}\lambda(s,\omega)ds\right]\right] = 0$, respectively.

Example 2.6.4. We consider the initial value problems (2.1.18), (2.1.19) and (2.1.20). We assume that $\widehat{A}(s) = E[A(s,\omega)]$. Setting

$V(t, x, \omega) = \|x\|^2$ and following the discussion in Example 2.1.1, one can arrive at

$$\|y(t,\omega) - \bar{x}(t)\|^2 =$$
$$\|x(t,\omega) - x(t)\|^2 + \int_{t_0}^{t} \mathcal{D}V(s, x(t,s,y(s,\omega)) - x(t,s,x(s)), \omega)ds \quad (2.6.16)$$

where

$$\mathcal{D}V(s, x(t,s,y) - x(t,s,z), \omega) =$$
$$2(y-z)^T \Phi^T(t,s)\Phi(t,s)[A(s,\omega) - \widehat{A}(s)]y.$$

In this case, it is obvious that the hypotheses (i) and (ii) of Theorem 2.6.2 are satisfied. To verify the remaining hypotheses of Theorem 2.6.2, we assume that $\Phi(t, t_0)$ satisfies the following relation

$$\|\Phi(t,t_0)\| \leq K \exp[-\gamma(t - t_0)], \qquad t \geq t_0 \qquad (2.6.17)$$

where γ is some positive real number. Using (2.6.16), we compute

$$\begin{aligned}
&|\mathcal{D}V(s, x(t,s,y) - x(t,s,z), \omega)| \\
&\leq 2K^2\|y - z\| \exp[-2\gamma(t-s)]\|A(s,\omega) - \widehat{A}(s)\|\|y\| \\
&\leq K^2 \exp[-2\gamma(t-s)]\|A(s,\omega) - \widehat{A}(s)\|(\|y-z\|^2 + \|y\|^2) \\
&\leq K^2 \exp[-2\gamma(t-s)]\|A(s,\omega) - \widehat{A}(s)\|(3\|y-z\|^2 + 2\|z\|^2).
\end{aligned}$$

By setting $\eta(t-s) = 3K^2 \exp[-2\gamma(t-s)]$, $\lambda_1(s,\omega) = \lambda_2(s,\omega) = \|A(s,\omega) - \widehat{A}(s)\| = \lambda(s,\omega)$, the above relation reduces to

$$|\mathcal{D}V(s, x(t,s,y) - x(t,s,z), \omega)| \leq$$
$$\eta(t-s)\lambda(s,\omega)\Big(V(s, y-z, \omega) + V(s, z, \omega)\Big).$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

This verifies the hypothesis (iii) of Theorem 2.6.2, provided that λ is sample integrable. Further, assume that

$$E\left[\exp\int_{t_0}^{t}\lambda(s,\omega)ds\right] \leq \exp\left[\int_{t_0}^{t}\widehat{\lambda}(s)ds\right] \tag{2.6.18}$$

for some integrable function $\widehat{\lambda}$ on R_+ into R_+, and

$$\lim_{t\to\infty}\sup\left[\frac{1}{t-t_0}\int_{t_0}^{t}\widehat{\lambda}(s)ds\right] < 2\gamma \tag{2.6.19}$$

where γ is as defined in (2.6.17). The hypotheses (iv) and (v) of Theorem 2.6.2 follow from relation (2.6.16) and definition of $\eta(t-s) = 3K^2\exp[-2\gamma(t-s)]$ with $\beta(t-s) = \eta(t-s)$, $\alpha(r) = r$, $p = 2$. Therefore, from (2.6.17) and (2.6.18), the relative asymptotic stability of (2.1.18) and (2.1.19) (or (2.1.20)) follows by the application of Theorem 2.6.2.

2.7. APPLICATIONS TO POPULATION DYNAMICS

In this section, we discuss mathematical models in population dynamics. In particular, we consider mathematical models of population growth of competing species as a prototype model of our analysis. This model is based on a number of simplified assumptions, as follows:

(i) The density of a species, that is, the number of individuals per unit area, can be adequately represented by a single variable. This ignores age differences, and also differences of sex and genotype.

(ii) Crowding affects all population members equally. This is unlikely to be true if the members of the species occur in clumps rather than being evenly distributed throughout the available space.

(iii) The effects of interactions within and between species are instantaneous. This means that there is no delayed action on the dynamics of the population.

(iv) Abiotic environmental factors are not constant.

(v) Population growth rate is density-dependent even at the lowest densities. It may be more reasonable to suppose that there is some threshold density below which individuals do not interfere with one another.

(vi) The females in a sexually reproducing population always find mates, even though the density may be low.

(vii) The intra- and inter-specific interactions are perturbed by random environmental disturbances.

The assumptions relative to the density-dependency and crowding effects reflect the fact that the growth of any species in a restricted environment must eventually be limited by a shortage of resources. We consider an n-species community model under random environmental fluctuations living together and competing with each other for the same limiting resources. Under the assumptions (i)–(vii), a mathematical model of population growth of n competing species is described by

$$N_i' = N_i \left(a_i(\omega) - b_{ii}(\omega) N_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} b_{ij}(\omega) N_j \right) \tag{2.7.1}$$

$$N_i(0) = N_{i0}, \qquad \text{for } i = 1, 2, \ldots, n,$$

where N_i is a density of the i^{th} species in the community; a_i and $b_{ij} \in R[\Omega, R]$; for $i = 1, 2, \ldots, n$, $a_i(\omega)$, $b_{ii}(\omega)$ are positive random variables; random variables $b_{ij}(\omega)$ have any sign for $i \neq j$. Any arbitrary sign of $b_{ij}(\omega)$ for $j \neq i$ allows us a greater flexibility about the interactions between the i^{th} and j^{th} species in the community. For example, competition $(-, -)$, predator-prey $(+, -)$, and mutualism

or symbiosis $(+,+)$, that is, for $i \neq j$, $b_{ij}(\omega)$ and $b_{ji}(\omega)$ are negative, $b_{ij}(\omega) > 0$ and $b_{ji}(\omega) < 0$, and $b_{ij}(\omega)$ and $b_{ji}(\omega)$ are positive, respectively. These equations are derived from the Verhulst-Pearl logistic equation

$$\frac{dN_i}{dt} = N_i\big(a_i(\omega) - b_{ii}(\omega)N_i\big) \tag{2.7.2}$$

for $i = 1, 2, \ldots, n$ by including the additional terms $-b_{ij}(\omega)N_j$ for $i, j = 1, 2, \ldots, n$ and $i \neq j$ to describe the inhibiting effects of each species on its competitor. The logistic equation is best regarded as a purely descriptive equation. The important features of (2.7.2) are: (a) the species increase exponentially whenever they are isolated and rare, and (b) they approach their equilibrium without oscillations in the absence of its competitor. There are several examples in the literature that show the close correspondence between the growth of an actual laboratory population and the theoretical curve associated with the solution of differential equation (2.7.2),

$$N_i(t) = a_i(\omega)/b_{ii}(\omega)[1+\exp[-a_i(\omega)t](a_i(\omega)/b_{ii}(\omega) - N_i(0))/N_i(0)]^{-1}. \tag{2.7.3}$$

For instance, Gause [38] describes the growth of a culture of *Paramecium caudatum*, Lotka [90], an experimental population of *Drosophila* and a colony of bacteria, and Odum [97], the growth of yeast in a culture.

In (2.7.1), for $i = 1, 2, \ldots, a_i(\omega)N_i$ can be interpreted as the potential rate of increase of an i^{th} species of density N_i, that is, the rate at which the i^{th} species would grow if the resources were unlimited and intra/inter-specific effects are neglected. Here $a_i(\omega)$ is the intrinsic rate of natural increase of the i^{th} species. $a_i(\omega)/b_{ii}(\omega) = k_i(\omega)$ is referred as the carrying capacity of the i^{th} species. From this, (2.7.2)

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

can be rewritten as

$$\frac{dN_i}{dt} = a_i(\omega)N_i(1 - N_i/k_i). \qquad (2.7.4)$$

We observe that the per capita growth rate $(dN_i/dt)/N_i$ will be negative or positive depending on the population density $N_i > k_i$ or $N_i < k_i$. Thus the k_i determines the saturation level of population densities.

Suppose the equilibrium points of (2.7.1) are given by $\alpha_i(\omega)$, satisfying the equations,

$$\alpha_i(\omega)\left(a_i(\omega) - \sum_{j=1}^{n} b_{ij}(\omega)\alpha_j(\omega)\right) = 0; \qquad i = 1, 2, 3, \ldots, n.$$

Assume that $\alpha_i(\omega) > 0$, $i = 1, 2, 3, \ldots, n$. Then the above equation reduces to

$$a_i(\omega) - \sum_{j=1}^{n} b_{ij}(\omega)\alpha_j(\omega) = 0; \qquad i = 1, 2, 3, \ldots, n. \qquad (2.7.5)$$

Consider the transformation

$$y_i = N_i - \alpha_i(\omega), \qquad i = 1, 2, 3, \ldots, n.$$

Using this transformation, (2.7.1) reduces to

$$\begin{aligned} y_i' &= -b_{ii}(\omega)y_i(\alpha_i(\omega) + y_i) - \sum_{j=1, j\neq i}^{n} b_{ij}(\omega)y_jy_i - \sum_{j=1, j\neq i}^{n} b_{ij}(\omega)\alpha_i(\omega)y_j \\ &= -b_{ii}(\omega)y_i\alpha_i(\omega) - \sum_{j=1}^{n} b_{ij}(\omega)y_jy_i - \sum_{j=1, j\neq i}^{n} b_{ij}(\omega)\alpha_i(\omega)y_j. \end{aligned} \qquad (2.7.6)$$

Considering $-\sum_{j=1}^{n} b_{ij}(\omega)y_jy_i$ as the perturbation term, the system (2.7.6) can be written as

$$y' = B(\omega, \alpha(\omega))y - Y(y)B(\omega)y, \qquad y(t_0) = N_0 - \alpha(\omega), \qquad (2.7.7)$$

where $Y(y) = \mathrm{Diag}(y_1, y_2, \ldots, y_n)$; $B(\omega) = (b_{ij}(\omega))_{n\times n}$; and

$$B(\omega, \alpha(\omega)) = \begin{bmatrix} -b_{11}(\omega)\alpha_1(\omega) & -b_{12}(\omega)\alpha_1(\omega) & \cdots & -b_{1n}(\omega)\alpha_1(\omega) \\ -b_{21}(\omega)\alpha_2(\omega) & -b_{22}(\omega)\alpha_2(\omega) & \cdots & -b_{2n}(\omega)\alpha_2(\omega) \\ \vdots & \vdots & \vdots & \vdots \\ -b_{n1}(\omega)\alpha_n(\omega) & -b_{n2}(\omega)\alpha_n(\omega) & \cdots & -b_{nn}(\omega)\alpha_n(\omega) \end{bmatrix},$$

$$\alpha(\omega) = \begin{bmatrix} \alpha_1(\omega) \\ \alpha_2(\omega) \\ \vdots \\ \alpha_n(\omega) \end{bmatrix}, \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

By setting $(\bar{b}_{ij})_{n\times n} = (E[b_{ij}])_{n\times n}$, and $\bar{a}_i = E[a_i]$, we write $\bar{B} = E[B]$, $\bar{a} = E[a]$ and $\bar{B}(\alpha^*) = (-\bar{b}_{ij}\alpha_i^*)_{n\times n}$. The smooth system corresponding to the system described by (2.7.1) is represented by

$$\bar{N}_i' = \bar{N}_i \left(\bar{a}_i - \sum_{j=1}^{n} \bar{b}_{ij}\bar{N}_j \right), \qquad \bar{N}(t_0) = E[N_0]. \tag{2.7.8}$$

By following the above argument, (2.7.8) can be written as

$$m' = \bar{B}(\alpha^*)m - M(m)\bar{B}m, \qquad m(t_0) = \bar{N}(t_0) - \bar{\alpha}^* \tag{2.7.9}$$

where α^* is determined by

$$\alpha_i^* \left(\bar{a}_i - \sum_{j=1}^{n} \bar{b}_{ij}\alpha_j^* \right) = 0, \qquad i = 1, 2, \ldots, n;$$

$\bar{B}(\alpha^*)$ and $\bar{B}$ as defined before and $M(m) = \mathrm{diag}(m_1, m_2, \ldots, m_n)$ matrix. The initial value problem (2.7.9) with random initial data is described by

$$x'(t,\omega) = \bar{B}(\alpha^*)x(t,\omega) - X(x)\bar{B}x(t,\omega), \qquad x(t_0,\omega) = x_0(\omega), \tag{2.7.10}$$

where $X(x) = \text{diag}(x_1, x_2, \ldots, x_n)$ matrix. Initial value problems (2.7.7), (2.7.9) and (2.7.10) can be rewritten as

$$y'(t,\omega) = \widehat{F}(t, y(t,\omega)) + R(t, y(t,\omega), \omega), \quad y(t_0,\omega) = y_0(\omega), \tag{2.7.11}$$

$$m' = \widehat{F}(t, m), \qquad m(t_0) = \bar{N}_0 - \bar{\alpha} = m_0 \tag{2.7.12}$$

and

$$x'(t,\omega) = \widehat{F}(t, x(t,\omega)), \qquad x(t_0,\omega) = x_0(\omega), \tag{2.7.13}$$

respectively, where $\widehat{F}(y,t) = \bar{B}(\alpha^*)y - Y(y)\bar{B}y$, $Y(y) =$ $\text{diag}(y_1, y_2, \ldots, y_n)$ and

$$R(t, y, \omega) = (B(\omega, \alpha(\omega)) - \bar{B}(\alpha^*))y - Y(y)(B(\omega) - \bar{B})y. \tag{2.7.14}$$

Let $y(t,\omega)$ be the solution process of (2.7.11) and let $\bar{x}(t) = x(t, t_0, z_0)$ be the solution process of either (2.7.12) or (2.7.13) depending on the initial data (t_0, m_0) or $(t_0, x_0(\omega))$. If $(t_0, x_0(\omega)) = (t_0, y_0(\omega))$, then $\bar{x}(t)$ is denoted by $x(t,\omega)$.

With this formulation, one can develop the results of Sections 2.1–2.6 for the Lotka-Volterra model of n-interacting species under random parametric disturbances.

In the following we present the result concerning stability of the zero equilibrium state of system (2.7.11). Of course, the stability of zero equilibrium state of (2.7.11) is the stability of the equilibrium state $\alpha(\omega)$ determined by (2.7.5). For this purpose we first establish the stability result concerning the trivial solution of system (2.7.12). The linear system corresponding to (2.7.12) is given by

$$z' = \bar{B}(\alpha^*)z, \qquad z(t_0) = z_0. \tag{2.7.15}$$

The solution of (2.7.15) can be written as

$$z(t, t_0, m_0) = \Phi(t, t_0) z_0 \tag{2.7.16}$$

where $\Phi(t, t_0)$ is the fundamental matrix solution of (2.7.15) and z_0 is either equal to m_0 or $y_0(\omega)$ (or $x_0(\omega)$).

The following result provides sufficient conditions for the stability of n-species community model under an ideal environment.

Theorem 2.7.1. *Assume that the fundamental matrix solution of* (2.7.15) *satisfies the inequality,*

$$\|\Phi^T(t,s)\Phi(t,s)\| \leq \exp[-2\beta(t-s)]; \qquad t_0 \leq s \leq t, \tag{2.7.17}$$

where β is a positive real number. Then the zero solution of (2.7.12) *is exponentially stable provided that $m_0 \in B_\rho$ some $\rho > 0$, where $B_\rho = [x \in R^n : \|x\| < \rho]$.*

Proof. Let $z(t, s, m(s))$ be the solution of (2.7.15) through $(s, m(s))$, where $m(s) = m(s, t_0, m_0)$ is the solution of (2.7.12) through (t_0, m_0).

Take $V(t, x) = \|x\|^2$, and apply the deterministic version of Theorem 2.1.1, we obtain

$$\begin{aligned} \frac{dV(s, z(t, s, m(s)))}{ds} &= 2z^T(t, s, m(s))\Phi(t, s)\big(\bar{B}(\alpha^*)m(s) - \widehat{F}(s, m(s))\big) \\ &= 2m^T(s)\Phi^T(t, s)\Phi(t, s)[-M(m(s))\bar{B}m(s)]. \end{aligned}$$

Using (2.7.17), the above equation becomes,

$$\frac{dV(s, z(t, s, m(s)))}{ds} \leq 2L\rho \exp[-2\beta(t-s)]\|m(s)\|^2, \text{ whenever } m \in B_\rho. \tag{2.7.18}$$

Integrating both sides of (2.7.18), one can obtain

$$V(t, z(t, t, m(t))) \leq V(t_0, z(t, t_0, m_0)) + 2\rho L \int_{t_0}^{t} \exp[-2\beta(t-s)]\|m(s)\|^2 ds,$$

whenever $m \in B_\rho$. Using the fact that $z(t,t,m(t)) = m(t)$ and $z(t,t_0,z_0) = z(t)$, the above inequality reduces to

$$\|m(t)\|^2 \leq \|z(t)\|^2 + 2\rho L \int_{t_0}^{t} \exp[-2\beta(t-s)]\|m(s)\|^2 ds, \quad \text{whenever } m \in B_\rho. \quad (2.7.19)$$

Multiplying both sides of (2.7.19) by $\exp[2\beta(t-t_0)]$, we have

$$\|m(t)\|^2 e^{2\beta(t-t_0)} \leq \|z(t)\|^2 e^{2\beta(t-t_0)} + 2\rho L \int_{t_0}^{t} \exp[2\beta(s-t_0)]\|m(s)\|^2 ds,$$

as long as $m \in B_\rho$. This together with the fact that $\|z(t)\|^2 \leq \|z_0\|^2 \exp[-2\beta(t-t_0)]$, one obtains

$$\|m(t)\|^2 e^{2\beta(t-t_0)} \leq \|z_0\|^2 + 2\rho L \int_{t_0}^{t} \exp[2\beta(s-t_0)]\|m(s)\|^2 ds, \quad (2.7.20)$$

whenever $m \in B_\rho$. Set $u(t) = \|m(t)\|^2 e^{2\beta(t-t_0)}$. From this, (2.7.20) becomes

$$u(t) \leq \|z_0\|^2 + 2\rho L \int_{t_0}^{t} u(s) ds, \qquad \text{whenever } m \in B_\rho.$$

By applying Gronwall's inequality, we can obtain

$$u(t) \leq \|z_0\|^2 \exp[2\rho L(t-t_0)], \qquad \text{as long as } m \in B_\rho,$$

which implies that

$$\|m(t)\|^2 \leq \|z_0\|^2 \exp[-2(\beta - \rho L)(t-t_0)], \qquad \text{as long as } m \in B_\rho. \quad (2.7.21)$$

We choose ρ such that $\rho < \beta/L$, and then choose z_0 such that $\|z_0\| < \rho$. From this choice of z_0, it is easy to see that $\|m(t)\| \leq \rho$. Hence $m(t) \in B_\rho$ for all $t \geq t_0$, whenever $\|z_0\| < \rho$. Thus (2.7.21) is valid for all $t \geq t_0$. From this, it is obvious that the zero solution of (2.7.12) is exponentially stable. This completes the proof of theorem.

Remark 2.7.1. One of the sufficient conditions, for Hypothesis (2.7.17) of Theorem 2.7.1, i.e.,

$$\|\Phi^T(t,s)\Phi(t,s)\| \le \exp[-2\beta(t-s)]; \qquad t_0 \le s \le t,$$

is the diagonal dominant condition for the matrix $\bar{B}(\alpha^*)$. For example, the column diagonal dominant condition for the matrix $\bar{B}$,

$$\bar{b}_{jj} > \sum_{i=1, i\neq j}^{n} |\bar{b}_{ij}|. \tag{2.7.22}$$

The column diagonal dominant condition for the matrix $\bar{B}(\alpha^*)$ or quasi-column dominant property of the matrix $\bar{B}$,

$$\bar{b}_{jj}\alpha_j^* > \sum_{i=1, i\neq j}^{n} |\bar{b}_{ij}|\alpha_i^*, \tag{2.7.23}$$

establishes the stability of linear system (2.7.15). Note that none of these conditions imply another. The stability condition (2.7.22) reflects the fact that the magnitude of the density-dependent effects of the i^{th} species exceeds the cross-interaction effects of the other species. The stability condition (2.7.23) reflects the fact that the intra specific growth-rate of the j^{th} species at the equilibrium state exceeds the total absolute turn-over-rate of the same species due to the presence of the other species at their saturation level. These statements are valid in the absence of randomness.

Corollary 2.7.1. *Assume that the hypotheses of* Theorem 2.7.1 *hold. Then the trivial solution process of* (2.7.13) *is exponentially stable with probability one. Moreover,*

$$\|x(t,\omega)\|^2 \le \|x_0(\omega)\|^2 \exp[-2(\beta-\rho L)(t-t_0)], \qquad t \ge t_0, \tag{2.7.24}$$

whenever $\|x_0(\omega)\| \le \rho$ *w.p. 1;* $\rho > 0$ *and* $\beta > \rho L$.

To obtain the sufficient conditions for the exponential stability of the trivial solution process of (2.7.11), we need to observe certain qualitative properties of the solution of smooth problem (2.7.13). We note that

$$\frac{\partial \widehat{F}}{\partial y}(t,y) = \bar{B}(\bar{\alpha}) - Y(y)\bar{B}_0 \tag{2.7.25}$$

where

$$\bar{B}_0 = \begin{bmatrix} 2\bar{b}_{11} & \bar{b}_{12} & \cdots & \bar{b}_{1n} \\ \bar{b}_{21} & 2\bar{b}_{22} & \cdots & \bar{b}_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{b}_{n1} & \bar{b}_{n2} & \cdots & 2\bar{b}_{nn} \end{bmatrix}$$

and hence

$$\begin{aligned} \frac{\partial x}{\partial x_0}(t,t_0,x_0(\omega)) &= \Phi(t,t_0,x_0(\omega)), \\ x(t,\omega) &= \int_0^1 \Phi(t,t_0,rx_0(\omega))dr\, x_0(\omega). \end{aligned} \tag{2.7.26}$$

From (2.7.11), (2.7.13), (2.7.26), taking $V(t,x) = \|x\|^2$ and the application of Theorem 2.1.1, we have

$$\begin{aligned} \|y(t,\omega)\|^2 &= \|x(t,\omega)\|^2 \\ &+ \int_{t_0}^t 2x^T(t,s,y(s,\omega))\Phi(t,s,y(s,\omega))R(s,y(s,\omega),\omega)ds. \end{aligned} \tag{2.7.27}$$

Now one can imitate the argument used in the proof of Theorem 2.7.1 and can conclude that the trivial solution process of (2.7.11) is exponentially stable with probability one if

$$\beta > \rho(\|B(\omega,\alpha(\omega)) - \bar{B}(\alpha)\| + \|B(\omega) - \bar{B}\| + L), \qquad \text{w.p. } 1 \tag{2.7.28}$$

for some $\rho > 0$ and β as defined in (2.7.17). Moreover, from (2.7.24) with $x_0(\omega) = y_0(\omega)$

$$\begin{aligned} \|y(t,\omega)\|^2 \le \|y_0(\omega)\|^2 \exp\Big[-2\big[\beta - \rho\big(\|B(\omega,\alpha(\omega)) - \bar{B}(\bar{\alpha})\| \\ + \|B(\omega) - \bar{B}\| + L\big)\big](t-t_0)\Big], \qquad t \ge t_0. \end{aligned} \tag{2.7.29}$$

Further we remark that the exponential mean square stability of the trivial solution of (2.7.11) can be discussed from (2.7.29) with some conditions on $y_0(\omega)$, $B(\omega, \alpha(\omega)) - \bar{B}(\bar{\alpha})$ and $B(\omega) - \bar{B}$. The details are left to the reader.

In order to further gain the insight to the preceeding discussion, we further discuss a system (2.7.1) with $n = 1$ (Logistic Growth Model).

$$\frac{dN}{dt} = N(a(t,\omega) - b(t,\omega)N), \qquad N(t_0,\omega) = N_0 \tag{2.7.30}$$

where $a, b \in M[R_+ \times R, R[\Omega, R_+]]$, $k(\omega) = a(t,\omega)/b(t,\omega)$. We observe that the equilibrium state $k(\omega)$ is positive random variable with positive mean $\bar{k} = E[k(\omega)]$. By using the transformation $y = k(\omega) - N$, (2.7.30) reduces to

$$y'(t,\omega) = -b(t,\omega)(k(\omega) - y(t,\omega))y(t,\omega), \qquad y(t_0,\omega) = k(\omega) - N_0. \tag{2.7.31}$$

Assume that $\bar{a}(t) = E[a(t,\omega)]$, $\bar{b}(t) = E[b(t,\omega)]$ and the Logistic Growth Model in the absence of randomness is described by

$$m' = -\bar{b}(t)(k^* - m)m, \qquad m(t_0) = k^* - E[N_0] = m_0, \tag{2.7.32}$$

where $k^* = \bar{a}(t)/\bar{b}(t)$. From (2.7.32), (2.7.31) can be rewritten as

$$y'(t,\omega) = -\bar{b}(t)(k^* - y(t,\omega))y(t,\omega) + R(t, y(t,\omega), \omega), \quad y(t_0,\omega) = y_0(\omega), \tag{2.7.33}$$

where

$$R(t, y(t,\omega), \omega) = -y(t,\omega)\Big(\Delta a(t,\omega) - \Delta b(t,\omega)y(t,\omega)\Big); \tag{2.7.34}$$

where $\Delta b(t,\omega) = b(t,\omega) - E[b(t,\omega)]$, $\Delta a(t,\omega) = a(t,\omega) - E[a(t,\omega)]$.

We note that the solution of (2.7.32) is given by

$$m(t) = k^* m_0 \exp\left[-\int_{t_0}^{t} \bar{a}(s)ds\right] \left(k^* - m_0 + m_0 \exp\left[-\int_{t_0}^{t} \bar{a}(s)ds\right]\right)^{-1}, \tag{2.7.35}$$

for $t \geq t_0$. Moreover, it is easy to see that

$$\begin{aligned} \frac{\partial m(t)}{\partial m_0} &= \Phi(t, t_0, m) \\ &= k^{*2} \exp\left[-\int_{t_0}^{t} \bar{a}(s)ds\right] \left(k^* - m_0 + m_0 \exp\left[-\int_{t_0}^{t} \bar{a}(s)ds\right]\right)^{-2}; \end{aligned} \tag{2.7.36}$$

$$\begin{aligned} |m(t)| &\leq k_1 |m_0| \exp\left[-\int_{t_0}^{t} \bar{a}(s)ds\right] \\ \text{and} \quad |\Phi(t, t_0, m_0)| &\leq k_2 \exp\left[-\int_{t_0}^{t} \bar{a}(s)ds\right], \end{aligned} \tag{2.7.37}$$

where $k_1 = k^*/k_1^*$, $k_2 = (k^*/k_1^*)^2$, $k_1^* = \min\{k^*, E[N_0]\}$. We take $V(t, x) = |x|^2$ and compute

$$V(s, x(t, s, y(s, \omega))) = \frac{1}{2} |x(t, s, y(s, \omega))|^2, \tag{2.7.38}$$

where $y(s, \omega)$ is the solution process of (2.7.33) for $t_0 \leq s \leq t$ and $x(t, s, y, (s, \omega))$ is the solution process of the differential equation in (2.7.32) with the initial conditions $(s, y(s, \omega))$. From (2.7.38), (2.7.35) and (2.7.36) we obtain

$$\begin{aligned} D^+_{(2.7.33)} V(s, x(t, s, y(s, \omega))) = \frac{d}{ds} \Bigg(&\frac{k^{*2}}{2} y^2(s, \omega) \exp\left[-2 \int_s^t a(u, \omega)du\right] \\ &\times \left(k^* - y(s, \omega) + y(s, \omega) \exp\left[-\int_s^t a(u, \omega)du\right]\right)^{-2} \Bigg) \end{aligned}$$

$$= k^{*3} y(s,\omega) \exp\left[-2\int_s^t a(u,\omega)du\right]$$
$$\times \left(k^* - y(s,\omega) + y(s,\omega)\exp\left[-\int_s^t a(u,\omega)du\right]\right)^{-3}$$
$$\times R(s, y(s,\omega), \omega)$$
$$= -2|x(t,s,y(s,\omega))|^2 k^* \Big(\Delta a(t,\omega) - \Delta b(t,\omega) y(s,\omega)\Big)$$
$$\times \left(k^* - y(s,\omega) + y(s,\omega)\exp\left[-\int_s^t a(u,\omega)du\right]\right)^{-1}$$
$$\leq \Delta\nu(s,\omega) V(s, x(t,s,y(s,\omega))) \quad (2.7.39)$$

$t_0 \leq s \leq t_0$ whenever $|y(s,\omega)| \leq \rho$ w.p. 1 and $\Delta\nu \geq 0$ for some $\rho > 0$. Here the comparison equation is

$$u' = \Delta\nu(s,\omega)u, \qquad u(t_0,\omega) = u_0(\omega), \tag{2.7.40}$$

where $\Delta\nu(s,\omega) = 2\Big(|\Delta a(s,\omega)| + \rho|\Delta b(s,\omega)|\Big)$. Here $r(s,t_0,u_0) = u_0 \exp\left[\int_{t_0}^s \Delta\nu(s,\omega)ds\right]$ with $u_0 = \frac{1}{2}|x(t,t_0,y_0(\omega))|^2$. Thus, by the application of Theorem 2.2.1, we get

$$|y(t,\omega)|^2 \leq |x(t,\omega)|^2 \exp\left[\int_{t_0}^t \Delta\nu(s,\omega)ds\right], \qquad t \geq t_0, \tag{2.7.41}$$

where $\Delta\nu(s,\omega) = \big(|\Delta a(s,\omega)| + |\Delta b(s,\omega)|\rho\big)$, whenever $|y(t,\omega)| \leq \rho$. This together with the argument used in the proof of Theorem 2.7.1, one can conclude that (2.7.41) is valid whenever $|y_0(\omega)| \leq \rho$. From (2.7.41) one can easily establish the stability in the mean-square as well as with probability one.

2.8. NUMERICAL EXAMPLES

To show the scope and usefulness of the results in Section 2.5, we present some numerical examples. We develop a computer code to generate samples of random differential equations.

$$\dot{y}(t,\omega) = a(t,\omega)y(t,\omega), \quad y(t_0,\omega) = y_0(\omega), \tag{2.8.1}$$

$$\dot{m}(t) = E(a(t,\omega))m(t), \quad E(y_0(\omega)) = m_0, \tag{2.8.2}$$

and

$$\dot{y}(t,\omega) = -a(t,\omega)y^3(t,\omega), \quad y(t,\omega) = y_0(\omega) \tag{2.8.3}$$

$$\dot{m}(t) = -E(a(t,\omega))m^3(t), \quad E(y_0(\omega)) = m_0. \tag{2.8.4}$$

For our calculations we have taken $a(t,\omega)$ to be uniformly distributed in $(0, 0.5)$. We use subroutines in IMSL Scientific Library to generate random variables and to evaluate the solution of the differential equations (2.8.1) and (2.8.2) at $t = 0.1, 0.2, \ldots, 1.0$, when $y(t_0,\omega) = .25, .5, .75$ and $m(t_0) = .25, .5, .75$. Using these numerical results we calculate $E|y(t,\omega) - m(t)|$ and $E|y(t,\omega) - m(t)|^2$. Furthermore, we get

$$\begin{aligned} E(\|y(t,\omega) - m(t)\|^p) \leq \\ 2^p\Bigg[E^{1/2}\|x_0(\omega) - m_0\|^{2p}E^{1/2}\left(\exp\left(2p\int_{t_0}^t \alpha(s,\omega)ds\right)\right) \\ \times\exp\left(p\int_{t_0}^t \Lambda(s)ds\right) + \|m_0\|^p\left\{\int_{t_0}^t \exp\left(2\int_s^t \Lambda(u)du\right)ds\right\}^{p/2} \\ \times E\left\{\int_{t_0}^t \xi(s,\omega)^2\exp\left(2\int_s^t \alpha(z,\omega)dz\right)ds\right\}^{p/2}\Bigg]. \end{aligned} \tag{2.8.5}$$

The upper bound in (2.8.5) is calculated for (2.8.1) and (2.8.2). All of these numerical results justify our claim that error estimates obtained via the differential inequality are better than that were obtained via the integral inequality. Some of these results are presented in the figures.

In Figures 2.8.1 through 2.8.4, x-axis represents t values and y represents upper bounds of comparison theorem and variation of constant formula which are presented in Tables 2.8.1 to 2.8.4, respectively. In the figures

_________ comparison theorem upper bound when $m_0 = .25$

____ ____ comparison theorem upper bound when $m_0 = .5$

____ _ ____ comparison theorem upper bound when $m_0 = .75$

$\cdots\cdots\cdots$ variation of constant formula upper bound where no control over m_0.

Figures 2.8.5 and 2.8.6 represent the mean of the solution and solution of the mean of (2.8.1) and (2.8.2), respectively.

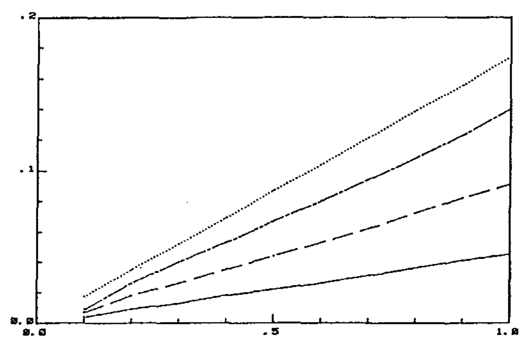


Figure 2.8.1 Upper bound of $E|y - \widehat{y}|$, $\dot{y} = ay$

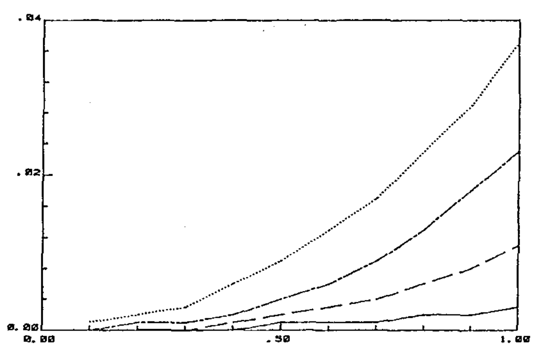


Figure 2.8.2 Upper bound of $E|y - \widehat{y}|^2$, $\dot{y} = ay$

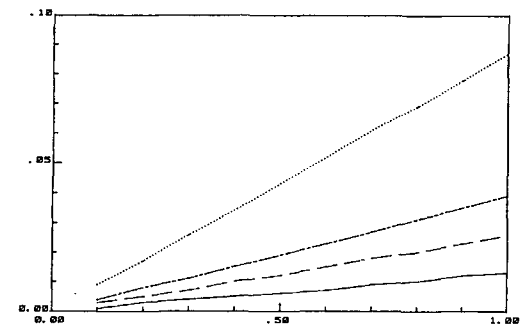


Figure 2.8.3 Upper bound of $E|y - \widehat{y}|$, $\dot{y} = -ay^3$

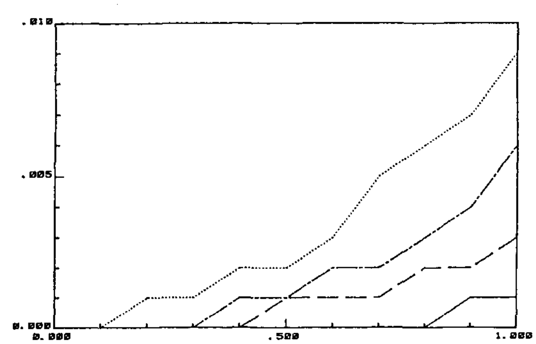


Figure 2.8.4 Upper bound of $E|y - \widehat{y}|^2$, $\dot{y} = -ay^3$

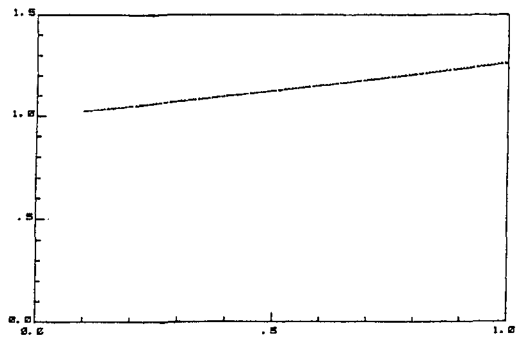


Figure 2.8.5 Mean of the Solutions - Solution of the Mean, $\dot{y} = ay$

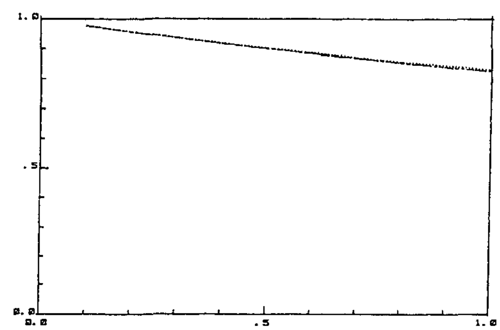


Figure 2.8.6 Mean of the Solutions - Solution of the Mean, $\dot{y} = -ay^3$

Table 2.8.1 $E|y(t,\omega) - m(t)|$ and analytic upper bounds for (2.8.1)

t	$E\|y(t,\omega) - m(t)\|$, m_0 = .25	.5	.75	Upper Bound: Comparison Principle, m_0 = .25	.5	.75	Upper Bound: Variation of Constant Formula
0.1	.0022	.0045	.0131	.0044	.0067	.0087	.0174
0.2	.0046	.0091	.0137	.0088	.0176	.0264	.0348
0.3	.0070	.0140	.0210	.0133	.0265	.0398	.0522
0.4	.0096	.0191	.0287	.0178	.0355	.0533	.0696
0.5	.0122	.0245	.0367	.0223	.0447	.0670	.0870
0.6	.0151	.0301	.0452	.0269	.0539	.0808	.1044
0.7	.0180	.0360	.0540	.0316	.0632	.0948	.1218
0.8	.0211	.0421	.0632	.0363	.0726	.1089	.1392
0.9	.0243	.0485	.0728	.0411	.0821	.1232	.1566
1.0	.0276	.0552	.0828	.0459	.0917	.1376	.1740

Table 2.8.2 $E|y(t,\omega) - m(t)|^2$ and analytic upper bounds for (2.8.1)

t	$E\|y(t,\omega) - m(t)\|^2$, m_0 = .25	.5	.75	Upper Bound: Comparison Principle, m_0 = .25	.5	.75	Upper Bound: Variation of Constant Formula
0.1	.0000	.0000	.0001	.0000	.0001	.0001	.0004
0.2	.0000	.0001	.0002	.0001	.0002	.0006	.0015
0.3	.0001	.0002	.0005	.0001	.0006	.0013	.0033
0.4	.0001	.0004	.0010	.0003	.0011	.0025	.0058
0.5	.0002	.0007	.0016	.0005	.0019	.0042	.0091
0.6	.0003	.0011	.0025	.0007	.0029	.0064	.0131
0.7	.0004	.0016	.0035	.0010	.0042	.0094	.0178
0.8	.0005	.0021	.0048	.0015	.0058	.0131	.0233
0.9	.0007	.0028	.0064	.0020	.0079	.0177	.0294
1.0	.0009	.0037	.0083	.0026	.0104	.0234	.0363

Table 2.8.3 $E|y(t,\omega) - m(t)|$ and analytic upper bounds for (2.8.2)

	$E\|y(t,\omega) - m(t)\|$			Upper Bound			
				Comparison Principle			
	m_0			m_0			Variation of
t	.25	.5	.75	.25	.5	.75	Constant Formula
0.1	.0001	.0011	.0035	.0012	.0025	.0037	.0087
0.2	.0003	.0021	.0068	.0025	.0050	.0075	.0174
0.3	.0004	.0031	.0098	.0038	.0076	.0114	.0261
0.4	.0005	.0041	.0126	.0051	.0102	.0152	.0348
0.5	.0007	.0050	.0152	.0064	.0128	.0191	.0435
0.6	.0008	.0059	.0177	.0077	.0154	.0231	.0522
0.7	.0009	.0068	.0200	.0090	.0181	.0271	.0609
0.8	.0011	.0076	.0221	.0104	.0207	.0311	.0696
0.9	.0012	.0084	.0241	.0117	.0235	.0352	.0783
1.0	.0013	.0092	.0259	.0131	.0262	.0393	.0870

Table 2.8.4 $E|y(t,\omega) - m(t)|^2$ and analytic upper bounds for (2.8.2)

	$E\|y(t,\omega) - m(t)\|^2$			Upper Bound			
				Comparison Principle			
	m_0			m_0			Variation of
t	.25	.5	.75	.25	.5	.75	Constant Formula
0.1	.0000	.0000	.0000	.0000	.0000	.0000	.0001
0.2	.0000	.0000	.0001	.0000	.0001	.0001	.0004
0.3	.0000	.0000	.0001	.0000	.0001	.0003	.0008
0.4	.0000	.0000	.0002	.0001	.0003	.0006	.0015
0.5	.0000	.0000	.0003	.0001	.0005	.0010	.0023
0.6	.0000	.0000	.0004	.0002	.0007	.0016	.0033
0.7	.0000	.0001	.0005	.0003	.0010	.0023	.0045
0.8	.0000	.0001	.0006	.0004	.0015	.0033	.0058
0.9	.0000	.0001	.0007	.0005	.0020	.0044	.0074
1.0	.0000	.0001	.0008	.0006	.0026	.0058	.0091

2.9 NOTES AND COMMENTS

The determination of a deviation between the solution processes of a system of differential equations with random parameters and the solution processes of corresponding system of differential equations with deterministic parameters can be called as "stochastic versus deterministic" problem in the modeling of dynamic systems, Ladde and Sambandham [71]. The system description and a generalized variation of constants formula corresponding to the deterministic results of Ladde [57] are presented in Section 2.1. Theorems 2.1.1 and 2.1.2 are based on the result of Ladde and Sambandham [71, 72]. Corollary 2.2.1 is a well-known result of Ladde and Lakshmikantham [67]. Theorem 2.1.1 is based on the comparison result due to Ladde, Lakshmikantham and Sambandham [69], and it is an extension of deterministic result, namely, variational comparison theorem due to Ladde [57] and also Ladde, Lakshmikantham and Leela [68]. Theorem 2.2.2 is adapted from Ladde, Lakshmikantham and Sambandham [69]. Corollary 2.2.1 is a well-known comparison theorem of Ladde [59]. Remarks 2.1.1, 2.2.1, Problem 2.1.3, and Corollary 2.2.2 are based on the ideas of Ladde [66]. The contents of Section 2.3 are based on the results of Kozin [53], Padgett, Schultz and Tsokos [99], Soong [104], and Soong and Chauang [102]. The contents of Sections 2.4 and 2.5 with regard to the generalized variation of parameters and variational comparison method are adapted from Ladde and Sambandham [72] and Ladde, Lakshmikantham and Sambandham [69], respectively. Theorem 2.4.1 is an extension of the deterministic result of Ladde [57] and Ladde, Lakshmikantham and Leela [68]. The results of Section 2.6 are new and are based on the deterministic results of Lakshmikantham

and Leela [83]. The material of Section 2.7 are based on the results of Ladde and Sathanathan [76]. See Capocelli and Ricciardi [13], Goel and Richter-Dyn [41], Levins [87], Levontin and Cohen [88], Lotka [90], May [91], Pielou [120], and Odom [97]. The numerical examples and figures in Section 2.8 are based on Ladde, Lakshmikantham and Sambandham [69]. Further related work, see Alevseev [1], Bharucha-Reid [6], Keller [48, 49], Ladde, Sambandham and Sathanathan [73], Ladde and Sathananthan [77, 78], Deo and Lakshmikantham [30] and Nashed and Engl [96].

CHAPTER 3: BOUNDARY VALUE PROBLEMS WITH RANDOM PARAMETERS

3.0 INTRODUCTION

This chapter is devoted to find a solution to the problem, "stochastic versus deterministic" in the modeling of dynamic processes that can be directly/indirectly described by boundary value problems with random parameters. Again, three major techniques for studying nonlinear boundary value problems are developed.

By introducing the idea of stochastic Green's function for a stochastic boundary value problems with random parameters, the existence of sample solution process of stochastic boundary value problem with random parameters is established in Section 3.1. Section 3.2 contains several comparison principles for stochastic boundary value problems with random parameters. A method of determining the probability distribution of the solution process of stochastic boundary value problems with random parameters is discussed in Section 3.3. The comparison theorems that are developed in Section 3.2, are utilized to establish the existence and uniqueness results for sample solution process of stochastic boundary value problems with random parameters in Section 3.4. By introducing the concept of the pth moment stability of the trivial solution process of stochastic boundary value problems with random parameters, the methods of Green's function and comparison are employed in Section 3.5 to study the pth moment stability of the stochastic boundary value problems with random parameters. Sections 3.6 and 3.7 deal with the error estimates and relative stability, respectively, of solution processes of stochastic boundary value problems with random parameters with respect

to the solution processes of the corresponding deterministic boundary value problems. Mathematical models for (i) slider bearing and right roller bearing in the theory of lubrication, and (ii) hanging cables under random perturbations illustrating the scope and usefulness of mathematical results are presented in Section 3.8. The scope and the significance of the mathematical results are further illustrated by numerical examples in Section 3.9.

3.1. GREEN'S FUNCTION METHOD

In this section, by using Green's function, we develop existence theorems for stochastic boundary value problems (SBVP for short) with random parameters.

We consider the stochastic boundary value problem (SBVP)

$$y'' = f(t, y, y', \omega), \qquad B_\mu(\omega) y(\mu, \omega) = b_\mu(\omega) \tag{3.1.1}$$

and corresponding deterministic boundary value problem (DBVP)

$$m'' = \widehat{f}(t, m, m'), \qquad \widehat{B}_\mu m(\mu) = \widehat{b}_\mu, \tag{3.1.2}$$

which is obtained by ignoring the randomness in the system described by (3.1.1).

In (3.1.1) and (3.1.2), $y, m \in R^n$;

$$B_\mu(\omega) y(\mu, \omega) = \alpha_\mu(\omega) y(\mu, \omega) + (-1)^{\mu+1} \beta_\mu(\omega) y'(\mu, \omega) \tag{3.1.3}$$

and

$$\widehat{B}_\mu m(\mu) = \widehat{\alpha}_\mu m(\mu) + (-1)^{\mu+1} \widehat{\beta}_\mu m'(\mu), \tag{3.1.4}$$

$\mu = 0, 1$; $\alpha_\mu, \beta_\mu \in R[\Omega, R]$, $b_\mu \in R[\Omega, R^n]$, $R[\Omega, R^n]$ denotes a collection of all n-dimensional random vectors defined on a complete

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

probability space $(\Omega, \mathcal{F}, P)$ into R^n and $n \geq 1$; $\widehat{\alpha}_\mu = E[\alpha_\mu, (\omega)]$, $\widehat{\beta}_\mu = E[\beta_\mu(\omega)]$, $\widehat{b}_\mu = E[b_\mu(\omega)]$, and E stands for expectation of a random variable; $f \in M[[0,1] \times R^n \times R^n, R[\Omega, R^n]]$ and $\widehat{f} \in C[[0,1] \times R^n \times R^n, R^n]$. We assume that

(H_1) $f(t,x,y,\omega)$ is $(\mathcal{F}^1 \otimes \mathcal{F}, \mathcal{F}^n)$-measurable for all $(x,y) \in R^n \times R^n$ and it is continuous with respect to $(x,y) \in R^n \times R^n$ for all $(t,\omega) \in J \times \Omega$, where $J = [0,1]$, $\mathcal{F}^1$ is a collection of Lebesgue-measurable subsets of J and $\mathcal{F}^n$ is the Borel σ-algebra of R^n;

(H_2) for $i = 0, 1$, α_i and β_i, satisfy

$$\alpha_i, \beta_i \geq 0 \qquad \text{and} \qquad \alpha_i + \beta_i > 0 \text{ with probability one.}$$

Let us introduce a definition of a sample solution process of (3.1.1).

Definition 3.1.1. A random process x defined on $J \times \Omega$ into R^n is said to a sample solution process of SBVP (3.1.1) if (i) it is sample continuously differentiable, (ii) $x'(t,\omega)$ is absolutely continuous, and (iii) $x(t,\omega)$ and $x'(t,\omega)$ satisfy boundary conditions and the system of stochastic differential equations in (3.1.1) almost everywhere with respect to the one-dimensional Lebesgue measure.

The proposed method for establishing the existence of a sample solution of (3.1.1) depends on its equivalent integral equation. For this purpose, we need the existence of stochastic Green's function of the corresponding SBVP

$$y'' = 0, \qquad B_\mu(\omega) y(\mu, \omega) = 0. \tag{3.1.5}$$

From (H_2), it is obvious that the SBVP (3.1.5) has a stochastic

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

Green's function $G(t,s,\omega)$ defined by

$$G(t,s,\omega)=\begin{cases} G_*(t,s,\omega)=\Big(\alpha_1(\omega)+\beta_1(\omega)-\alpha_1(\omega)t\Big)\Big(\alpha_0(\omega)s \\ \qquad +\beta_0(\omega)\Big)/D(\omega), \qquad 0\le s\le t\le 1 \\ G^*(t,s,\omega)=\Big(\alpha_0(\omega)t+\beta_0(\omega)\Big)\Big(\alpha_1(\omega)+\beta_1(\omega) \\ \qquad -\alpha_1(\omega)s\Big)/D(\omega), \qquad 0\le t\le s\le 1 \end{cases}$$

where

$$D(\omega)=-[\alpha_0(\omega)(\alpha_1(\omega)+\beta_1(\omega))+\beta_0(\omega)\alpha_1(\omega)].$$

Lemma 3.1.1. *Let us assume that* (H_1) *and* (H_2) *be satisfied. Furthermore,* $f(t,y(t,\omega),y'(t,\omega),\omega)$ *is sample Lebesgue integrable on* J*, whenever* $x(t,\omega)$ *is sample differentiable process and* $x'(t,\omega)$ *is jointly measurable process. Then SBVP* (3.1.1) *is equivalent to the stochastic integral equation*

$$y(t,\omega)=\psi(t,\omega)+\int_0^1 G(t,s,\omega)f(s,y(s,\omega),y'(s,\omega),\omega)ds, \qquad (3.1.6)$$

where $\psi(t,\omega)$ *is defined by* $b_0(\omega)\frac{\partial G}{\partial s}(t,0,\omega)-b_1(\omega)\frac{\partial G}{\partial s}(t,1,\omega)$ *and* $G(t,s,\omega)$ *is a stochastic Green's function defined by* (3.1.5).

Proof. From the definitions of $G(t,s,\omega)$ and $\psi(t,\omega)$ it is obvious that these random functions are product measurable. The proof of the lemma follows from the consequence of the method of variation of parameters. The details are left to the reader.

Before we prove the basis existence theorem we present a random fixed point theorem.

Theorem 3.1.1. *Let* X *be a separable Banach space and* $(\Omega,\mathcal{F},P)$ *be a complete probability space. Let* $r\in R[\Omega,R_+\backslash\{0\}]$ *and* F *be a mapping defined on* $X\times\Omega$ *into* X *such that*

(i) *$F(x, \omega)$ is a X-valued random variable for every $x \in X$, and it is continuous in x for every $\omega \in \Omega$.*

(ii) *$F(\bar{B}(r(\omega)), \omega)$ is compact for every $\omega \in \Omega$, where $\bar{B}(r(\omega), 0) = \{x \in X : \|x\| \leq r(\omega)\}$.*

Then there exists X-valued random variable such that $x(\omega) \in \bar{B}(r(\omega))$ and $F(x(\omega), \omega) = x(\omega)$ on Ω.

Proof. Let $B(\omega) = \bar{B}(r(\omega), 0)$ and R_ω be the retraction of X into $B(\omega)$ defined by: $R_\omega x = x$ for $\|x\| \leq r(\omega)$ and $R_\omega x = r(\omega)x/\|x\|$ for $\|x\| > r(\omega)$. Define a map, $\widetilde{F}$ on $X \times \Omega$ into X as $\widetilde{F}(x, \omega) = F(R_\omega x, \omega)$. Set $S(\omega) = \{x \in X : \widetilde{F}(x, \omega) = x\}$. Since $\widetilde{F}(x, \omega)$ is continuous in x for each $\omega \in \Omega$, $\widetilde{F}(X, \omega) = F(B(\omega), \omega)$ is relatively compact. By Schauder's fixed point theorem (Theorem A.3.1), $S(\omega)$ is non-empty and compact. Thus by the definition of $\widetilde{F}$, $S(\omega) = \{x \in B(\omega) : F(x, \omega) = x\}$. Furthermore, $\widetilde{F}(x, \omega)$ is X-valued random variable for every $x \in X$. To verify this statement let $\mathcal{B}$ be Borel σ-algebra of (X, d), where d is a metric induced by the norm, $\|\cdot\|$ on X. For $B \in \mathcal{B}$, we have

$$\begin{aligned}
\{\omega \in \Omega : \widetilde{F}(x, \omega) \in B\} = \Big(&\{\omega \in \Omega : F(x, \omega) \in B\} \\
&\cap \{\omega \in \Omega : r(\omega) \in [\|x\|, \infty]\}\Big) \\
&\cup \Big(\{\omega \in \Omega : F(r(\omega)x/\|x\|, \omega) \in B\} \\
&\cap \{\omega \in \Omega : r(\omega) \in [0, \|x\|]\}\Big),
\end{aligned}$$

where the first intersection is in $\mathcal{F}$, because $F(\cdot, x)$ and r are X and R-valued random variables, respectively; while the second intersection is also in $\mathcal{F}$, since r can be considered as the point-wise limit of a sequence of a step random variables.

We now apply Theorem A.3.2 to $S(\omega)$. We have shown that $S(\omega)$ is compact. Hence it is complete. Therefore, it is enough to show that $\rho(x_0, S(\cdot)) \in R[\Omega, R_+]$ for $x_0 \in X$. For this purpose, we need to show that the event $\{\omega \in \Omega : \rho(x_0, S(\omega)) \leq \alpha\} \in \mathcal{F}$ for an $\alpha \in R_+$. For $\alpha = 0$, $\{\omega \in \Omega : x_0 \in S(\omega)\} = \{\omega : \widetilde{F}(x_0, \omega) = x_0\} = \{\omega \in \Omega : \widetilde{F}(x_0, \omega) \in \{x_0\}\}$ is measurable because $\widetilde{F}(x_0, \omega)$ is X-valued random variable for every $x_0 \in X$. For $\alpha > 0$, $\rho(x_0, S(\omega)) \leq \alpha$ implies $S(\omega) \cap \bar{B}(\alpha, x_0) \neq \phi$. Since X is a separable Banach space, we have a countable dense subset $\{x_j \in X : j \in N\}$ of $\bar{B}(\alpha, x_0)$, and therefore $\{\omega \in \Omega : \rho(x_0, S(\omega)) \leq \alpha\} = \bigcap_{n \in N} \bigcup_{j \in N} \{\omega : \|x_j - \widetilde{F}(x_j, \omega)\| \leq \frac{1}{n}\} \in \mathcal{F}$, since $\widetilde{F}(X, \omega)$ is relatively compact and $\{\omega \in \Omega : \widetilde{F}(x_j, \omega) \in \bar{B}(\frac{1}{n}, x_j)\} \in \mathcal{F}$. Hence by the application of Theorem A.3.2, we conclude that S admits an $(\mathcal{F}, \mathcal{B})$-measurable selection, i.e., a map x defined on Ω into X such that for any $B \in \mathcal{B}$, $\{\omega : x(\omega) \in B\} \in \mathcal{F}$ and $x(\omega) \in S(\omega)$ for all $\omega \in \Omega$. This implies the conclusion of the theorem.

Now, we are ready to present an existence theorem for SBVP (3.1.1).

Theorem 3.1.2. *Let the hypotheses of* Lemma 3.1.1 *be satisfied. Further assume that f satisfies*

$$\|f(t, x, y, \omega)\| \leq M(t, \omega) \text{ for } (t, x, y, \omega) \in J \times R^n \times R^n \times \Omega, \quad (3.1.7)$$

where, $M \in M[J \times \Omega, R_+]$, and it is sample Lebesgue integrable on J. Then SBVP (3.1.1) *has a sample solution process on J.*

Proof. By setting $X = C^1[J, R^n]$, a set of continuously differentiable functions on J into R^n, we define

$$F(x, \omega)(t) = \psi(t, \omega) + \int_0^1 G(t, s, \omega) f(s, x(s), x'(s), \omega) ds \qquad \text{for } x \in X$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

and

$$r(\omega) = \max_{t\in J}\{\|\psi(t,\omega)\| + \int_0^1 |G(t,s,\omega)|M(s,\omega)ds,$$
$$\|\psi'(t,\omega)\| + \int_0^1 |G_t(t,s,\omega)|M(s,\omega)ds\}.$$

Note the proof of the theorem is a direct consequence of Theorem 3.1.1.

The boundedness assumption (3.1.7) on $f(t,x,y,\omega)$ in (x,y) seems to be restrictive. But under certain conditions on f, one can obtain an extension of Theorem 3.1.2. The following result deals with the case where $f(t,x,y,\omega)$ has at most a linear growth in x and y.

Theorem 3.1.3. *Let the assumptions of* Lemma 3.1.1 *be satisfied. Suppose that f satisfies the following linear growth condition as*

$$\|f(t,x,y,\omega)\| \le k_1(t,\omega)\|x\| + k_2(t,\omega)\|y\| + k_3(t,\omega), \tag{3.1.8}$$

for $(t,x,y,\omega) \in J \times R^n \times R^n \times \Omega$, where $k_i \in M[J\times\Omega, R_+]$ such that $k_i(t,\omega)$ are Lebesgue integrable relative to t for every $\omega \in \Omega$, $i \in \{1,2,3\}$. Further, suppose that the random linear operator, $H(\omega)$ defined on $C[J,R]\times C[J,R]$ into itself by

$$H(\omega)(u,v)(t) = \begin{bmatrix} \int_0^1 |G(t,s,\omega)|h(s,u(s),v(s),\omega)ds \\ \int_0^1 |G_t(t,s,\omega)|h(s,u(s),v(s),\omega)ds \end{bmatrix}^T \tag{3.1.9}$$

has a spectral radius less than one, where $h(s,u,v,\omega) = k_1(t,\omega)|u| + k_2(t,\omega)|v|$. Then SBVP (3.1.1) *has a sample solution on J.*

Proof. To prove the statement of the theorem, we first find the a-priori estimate for sample solutions of SBVP (3.1.1). For this purposes, let $x(t,\omega)$ be a sample solution process of SBVP (3.1.1) on J. From (3.1.6) we obtain

$$(\|x(t,\omega)\|, \|x'(t,\omega)\|) \le H(\omega)(\|x(\cdot,\omega)\|, \|x'(\cdot,\omega)\|)(t)$$
$$+ (M_0(\cdot,\omega), M_1(\cdot,\omega))(t)$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

with

$$M_0(t,\omega) = \|\psi(t,\omega)\| + \int_0^1 |G(t,s,\omega)| k_3(s,\omega) ds,$$

and

$$M_1(t,\omega) = \|\psi'(t,\omega)\| + \int_0^1 |G_t(t,s,\omega)| k_3(s,\omega) ds.$$

Hence

$$\begin{aligned}(\|x(t,\omega)\|, \|x'(t,\omega)\|) &\le (I - H(\omega))^{-1}(M_0(\cdot,\omega), M_1(\cdot,\omega))(t) \\ &\le \sum_{m=1}^{\infty} (H(\omega))^m (M_0(\cdot,\omega), M_1(\cdot,\omega))(t).\end{aligned}$$

This implies

$$\|x(t,\omega)\| \le M_2(\omega) \quad \text{and} \quad \|x'(t,\omega)\| \le M_2(\omega), \tag{3.1.10}$$

where

$$M_2(\omega) = \max_{t\in J} \left\| \sum_{m=1}^{\infty} (H(\omega))^m (M_0(\cdot,\omega), M_1(\cdot,\omega))(t) \right\|,$$

and $M_2 \in R[\Omega, R_+]$. Now, consider a process $\eta \in M[R^n \times \Omega, [0,1]]$ and $\eta(x,\omega)$ is continuous in x for each $\omega \in \Omega$ such that $\eta(x,\omega) = 1$ for $\|x\| \le M_2(\omega)$ and $\eta(x,\omega) = 0$ for $\|x\| \ge M_2(\omega) + 1$. Then the modified function of f is defined by

$$\widetilde{f}(t,x,y,\omega) = \eta(x,\omega)\eta(y,\omega) f(t,x,y,\omega). \tag{3.1.11}$$

It is easy to observe that $\widetilde{f}$ satisfies (H_1) and (3.1.7). Consequently, by the application of Theorem 3.1.2, the stochastic boundary value problem

$$y'' = \widetilde{f}(t,y,y',\omega), \qquad B_\mu(\omega) y(\mu,\omega) = b_\mu(\omega) \tag{3.1.12}$$

has a sample solution, $y(t,\omega)$. Furthermore, we note that $\widetilde{f}$ satisfies the linear growth condition (3.1.8). Therefore, any sample solution

process of (3.1.12) satisfies the a-priori estimate (3.1.10). From this, we have $\eta(y(t,\omega),\omega) = \eta(y'(t,\omega),\omega) = 1$ and therefore $y(t,\omega)$ is a solution process of (3.1.1) in view of the definition of f. The proof of the theorem is complete.

3.2. COMPARISON METHOD

It is well known that the comparison principle for the initial value problems has been very important principle in the theory of differential equations. The corresponding comparison results for deterministic boundary value problems have played an equally important role. In this section, we develop a very general comparison principle for systems of stochastic boundary value problems. These results will be used subsequently, throughout the chapter.

We consider the following SBVP

$$u_i'' = g_i(t,u,u_i',\omega), \qquad G_\mu(\omega)u(\mu,\omega) = r_\mu(\omega) \tag{3.2.1}$$

where $g_i \in M[J \times R^k \times R, R[\Omega, R]]$, $G_\mu(\omega)u(\mu,\omega) = \gamma_\mu(\omega)u(\mu,\omega) + (-1)^{\mu+1}\nu_\mu u'(\mu,\omega)$, $\mu = 0, 1, \gamma_\mu$, $\nu_\mu \in R[\Omega, R]$, $r_\mu \in R[\Omega, R^k]$, for $\mu = 0, 1$, and $i \in I_k$, $I_k = \{1, 2, \ldots, k\}$. We assume that g in (3.2.1) satisfies the Caratheodory type condition: (H_1) with (3.1.7)and boundary conditions in (3.2.1) satisfies condition (H_2).

We introduce the concepts of lower and upper sample solution processes of (3.2.1).

Definition 3.2.1. Let $\alpha \in M[J, R[\Omega, R^k]]$. The random process α is said to be lower sample solution process of (3.2.1), if $\alpha'(t,\omega)$ is absolutely continuous for every $\omega \in \Omega$, and it satisfies

$$\alpha_i''(t,\omega) \geq g_i(t,\alpha(t,\omega),\alpha_i'(t,\omega),\omega)$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

with a.e. (almost everywhere) on J and

$$G_\mu(\omega)\alpha(\mu, \omega) \leq r_\mu(\omega)$$

for every $\omega \in \Omega$ and $i \in I_k$.

The definition of an upper sample solution process of (3.2.1) can be given by replacing α by β and reversing the inequalities in Definition 3.2.1.

Before, we present a basic comparison principle, a concept of modified function relative to a given function is defined as follows.

Definition 3.2.2. Let $v \in M[J, R[\Omega, R^k]]$, and its derivative $v'(t, \omega)$ be absolutely continuous in t for every $\omega \in \Omega$. A random process $\widetilde{g}(t, y, y', \omega)$ is said to be modified process of $g(t, y, y', \omega)$ relative to v and random vector, $c \in R[\Omega, R_+^n]$ if

$$\widetilde{g}_i(t, y, y_i', \omega) = g_i(t, p(t, y, \omega), q_*(y_i', c_i(\omega)), \omega) + r_i(t, y, \omega), \quad (3.2.2)$$

where $p_i(t, y, \omega) = \max\{v_i(t, \omega), y_i\}$; $q_*(y_i', c_i(\omega)) = \max\{-c_i(\omega), \min\{y_i', c_i(\omega)\}\}$, and $r_i(t, y, \omega) = (y_i - p_i(t, y, \omega))/(1 + y_i^2)$, and c is defined by (3.1.7) relative to g, i.e., choose $c_i(\omega)$ such that $\int_0^1 M(s, \omega)ds < c_i(\omega)$ for all $i \in I_k$.

Now by assuming stronger regularity conditions on g in (3.2.1), we first present a basic comparison principle.

Theorem 3.2.1. *Assume that*

(i) $g \in M[J \times R^k \times R^k, R[\Omega, R^k]]$, $g(t, u, v, \omega)$ *satisfies hypothesis* (H_1) *and the quasi-monotone non-increasing property in* u *for each* $(t, v, \omega) \in J \times R^k \times \Omega$;

(ii) g *satisfies condition* (3.1.7);

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

(iii) $m \in M[J, R[\Omega, R^k]]$, *it is a lower sample solution process of SBVP* (3.2.1) *with boundary conditions satisfying* (H_2);

(iv) *for every* (t, u, ω), $g(t, u, v, \omega)$ *satisfies the locally Lipschitz condition in* v.

Then $m(t, \omega) \leq r(t, \omega)$ *for* $t \in J$, *where* $r(t, \omega)$ *is the maximal sample solution of* (3.2.1).

Proof. From hypotheses (H_1) and (H_2), condition (3.1.7) and the application of Theorem 3.1.2, the SBVP (3.2.1) has a sample solution process $u(t, \omega)$ on J. To prove the conclusion of the theorem, we consider the modified SBVP

$$u_i'' = \widetilde{g}_i(t, u, u_i', \omega), \qquad G_\mu(\omega)u(\mu, \omega) = r_\mu(\omega) \tag{3.2.3}$$

where $\widetilde{g}$ is defined with respect to $m(t, \omega)$ as in (3.2.2). We note that $\widetilde{g}$ satisfies the condition (3.1.7), and hypotheses (H_1), (H_2). Hence by the application of Theorem 3.1.2, the SBVP (3.2.3) has a sample solution process, $\widetilde{u}(t, \omega)$ on J. Now, it is enough to show that $m(t, \omega) \leq \widetilde{u}(t, \omega)$ on J. Because this would imply that

$$\widetilde{g}_i(t, \widetilde{u}(t, \omega), \widetilde{u}_i'(t, \omega), \omega) = g_i(t, \widetilde{u}(t, \omega), \widetilde{u}_i'(t, \omega), \omega) \qquad \text{for } i \in I_k.$$

In order to prove, $m(t, \omega) \leq \widetilde{u}(t, \omega)$ on J, we assume that it is false. Then there exists an $\omega \in \Omega$ and $i \in I_k$ such that $m_i(t, \omega) - \widetilde{u}_i(t, \omega)$ has a positive maximum, say at $t_0 \in J$. In the sequel, we shall omit ω. Suppose that $t_0 \in (0, 1)$. Then we can find positive numbers δ and ϵ, such that $m_i(t) \geq \widetilde{u}_i(t) + \epsilon$ with $m_i'(t_0) = \widetilde{u}_i'(t_0)$, on $J_\delta = [t_0, t_0 + \delta)$. This follows from the continuity of $\widetilde{u}_i$. From the above discussion, condition (ii) and the definition of $p(t, y, \omega)$ in (3.2.2), we have

$$m_i''(t) \geq g_i(t, m(t), m_i'(t), \omega)$$

and

$$\widetilde{u}_i''(t) = g_i(t, p(t, \widetilde{u}(t), \omega), \widetilde{u}_i'(t), \omega) + r_i(t, \widetilde{u}(t), \omega), \qquad \text{for a.e. } t \in J,$$

$$m_j(t) \leq p_j(t, \widetilde{u}(t), \omega) \qquad \text{for any } j \in I_k, t \in J$$

and

$$p_i(t, \widetilde{u}(t), \omega) = m_i(t) \qquad \text{for } t \in J_\delta.$$

From the above relations, continuity of $r_i(t, \widetilde{u}(t), \omega)$, the quasimonotone non-increasing property of $g(t, u, v, \omega)$ in u for fixed (t, v, ω), and setting $m_i'(t) = z_i(t)$ and $\widetilde{u}_i'(t) = v_i(t)$, we obtain

$$z_i'(t) \geq g_i(t, m(t), z_i(t), \omega)$$

and

$$v_i'(t) \leq g_i(t, m(t), v_i(t), \omega) - \epsilon_1 \qquad \text{for a.e. } t \in J_\delta \tag{3.2.4}$$

with

$$v_i(t_0) = z_i(t_0),$$

where ϵ_1 is some positive real number. Now, by applying Corollary A.3.1 and Theorem A.3.3, we obtain

$$\rho_i(t) \leq z_i(t) \text{ and } v_i(t) \leq r_i(t) \qquad \text{for } t \in J_\delta, \tag{3.2.5}$$

where $\rho_i(t)$ and $r_i(t)$ are the minimal and maximal solutions of

$$u_i' = g_i^*(t, u_i, \omega), \qquad u_i(t_0) = z_i(t_0), \tag{3.2.6}$$

where

$$g_i^*(t, u_i, \omega) = g_i(t, m(t), u_i, \omega).$$

From (iv), IVP (3.2.6) has unique solution, therefore $\rho_i(t) = r_i(t)$ on J_δ. This together with (3.2.5) implies that

$$v_i(t) \leq z_i(t) \qquad \text{for } t \in J_\delta. \tag{3.2.7}$$

From the definition of $v_i(t), z_i(t)$, and integrating (3.2.7), we obtain

$$m_i(t) - \widetilde{u}_i(t) \geq m_i(t_0) - \widetilde{u}_i(t_0) \text{ on } J_\delta.$$

This together with the fact that

$$m_i(t_0) - \widetilde{u}_i(t_0) \geq m_i(t) - \widetilde{u}_i(t) \text{ on } J_\delta$$

implies

$$m_i(t) = \widetilde{u}_i(t) + m_i(t_0) - \widetilde{u}_i(t_0) \text{ on } J_\delta$$

and hence

$$v_i(t) = z_i(t) \qquad \text{for } t \in J_\delta. \tag{3.2.8}$$

From (3.2.4) and (3.2.8) we get

$$v_i'(t) \leq z_i'(t) - \epsilon_1, \qquad \text{a.e. on } J_\delta.$$

By integrating the above inequality on J_δ, we obtain

$$\epsilon_1(t - t_0) \leq m_i'(t) - \widetilde{u}_i'(t) \qquad \text{for } t \in J_\delta.$$

This inequality contradicts to the fact that $m_i(t) - \widetilde{u}_i(t)$ has a positive maximum at $t_0 \in (0,1)$. Hence, there is no $\omega \in \Omega$ and $i \in I_k$ such that $m_i(t,\omega) - \widetilde{u}_i(t,\omega)$ can attain its positive maximum at $t_0 \in (0,1)$. This means that $t_0 \notin (0,1)$. Therefore, t_0 is either 0 or 1. Suppose that $t_0 = 0$. Hence

$$m_i(t) - \widetilde{u}_i(t) \leq m_i(0) - \widetilde{u}_i(0) \qquad \text{for small } t > 0.$$

This implies $m_i'(0) \leq \widetilde{u}_i'(0)$. From the boundary conditions, we also have

$$B_0(\omega)(m_i(0,\omega) - \widetilde{u}_i(0,\omega)) \leq 0$$

and hence

$$m_i'(0) = \widetilde{u}_i'(0)$$

in view of the (H_2). Now repeating the argument used in the case of $t_0 \in (0,1)$, we can conclude that $m_i(t) - \widetilde{u}_i(t)$ cannot attain its positive maximum at $t_0 = 0$. Similarly, one can easily see that $m_i(t) - \widetilde{u}_i(t)$ cannot attain its positive maximum at $t_0 = 1$. Hence there is no $\omega \in \Omega$ and $i \in I$ such that $m_i(t,\omega) - \widetilde{u}_i(t,\omega)$ attains its positive maximum on J. This means that $m(t,\omega) \leq \widetilde{u}(t,\omega)$ on J. Hence, by the definition of maximal solution

$$m(t,\omega) \leq r(t,\omega) \qquad \text{for } t \in J.$$

This completes the proof of the theorem.

Remark 3.2.1. A comparison result dual to Theorem 3.2.1 holds with appropriate modifications yielding $\rho(t,\omega) \leq m(t,\omega)$ on J, where $\rho(t,\omega)$ and $m(t,\omega)$ are the sample minimal solution process and an upper sample process of (3.2.1), respectively.

Corollary 3.2.1. *If condition* (3.1.7) *in* Theorem 3.2.1 *is replaced by* (3.1.8) *coupled with* (3.1.9), *the conclusion of theorem remains true.*

Remark 3.2.2. We note that the conditions of Theorem 3.2.1 and Corollary 3.2.1 are not used with their full force. For example, the role of conditions (ii) and (iv) of Theorem 3.2.1 can be optimized by replacing these conditions by the global Lipschitz condition in v and boundedness in u, i.e.,

$$|g_i(t,u,v_i,\omega) - g_i(t,u,\bar{v}_i,\omega)| \leq L_i^1(\omega)|\bar{v}_i - v_i|, \tag{3.2.9}$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

and

$$|g_i(t,u,0,\omega)| \le L_i^2(\omega) \tag{3.2.10}$$

for all $(t,u,\omega) \in J \times R^n \times \Omega$, $i \in I_k$, where $L_i^j \in R[\Omega, R_+]$ for $j = 1, 2$. The conclusion of Theorem 3.2.1 remains valid.

In order to further justify Remark 3.2.2, we need a concept of Nagumo condition, and some results relative to the concept.

Definition 3.2.3. Let $\alpha, \beta \in M[J, R[\Omega, R^k]]$, α and β be lower and upper sample solutions of (3.2.1) such that $\alpha(t,\omega) \le \beta(t,\omega)$. We say that g satisfies a Nagumo condition with respect to α and β, if there exists a random function $h \in M[R_+, R[\Omega, R_+^k]]$ such that $h(s,\omega)$ is continuous in s for every $\omega \in \Omega$

$$|g_i(t,u,v_i,\omega)| \le h_i(|v_i|,\omega) \qquad \text{for } (t,\omega) \in J \times \Omega \tag{3.2.11}$$

$\alpha(t,\omega) \le u \le \beta(t,\omega)$, $v_i \in R$, $i \in I_n$, and

$$\int_{\lambda_i(\omega)}^{N_i(\omega)} \frac{\tau d\tau}{h_i(\tau,\omega)} \ge \max_{t\in J} \beta_i(t,\omega) - \min_{t \in J} \alpha_i(t,\omega), \qquad \omega \in \Omega \tag{3.2.12}$$

for some $N \in R[\Omega, R_+^k]$ and

$$\lambda_i(\omega) = \max\{|\beta_i(0,\omega) - \alpha_i(1,\omega)|, |\beta_i(1,\omega) - \alpha_i(0,\omega)|\}.$$

Remark 3.2.3. We note that N in (3.2.12) exists if

$$\int_{\lambda_i(\omega)}^{\infty} \frac{\tau d\tau}{h_i(\tau,\omega)} = \infty \qquad \text{for each } \omega \in \Omega. \tag{3.2.13}$$

To verify this statement, we observe that

$$H_i(s,\omega) = \int_{\lambda_i(\omega)}^{s} \frac{\tau d\tau}{h_i(\tau,\omega)} \qquad \text{for } i \in I_k$$

is strictly increasing in s. We denote the right-hand side of (3.2.12) by $\nu_i(\omega)$, and define $N_i(\omega)$ by $H_i(N_i(\omega),\omega) = \nu_i(\omega) + 1$. For any $t \in R$, we consider

$$\{\omega \in \Omega : N_i(\omega) > t\} = \{\omega \in \Omega : H_i(N_i(\omega),\omega) > H_i(t,\omega)\}$$
$$= \{\omega \in \Omega : \nu_i(\omega) + 1 > H_i(t,\omega)\}.$$

Since $\{\omega \in \Omega : \nu_i(\omega) + 1 > H_i(t,\omega)\} \in \mathcal{F}$ for every $t \in R$, N_i is a random variable. Hence $N \in R[\Omega, R^k_+]$.

Here we present some sufficient conditions for obtaining a priori bounds for solutions and its derivative.

Lemma 3.2.1. *Assume that*

(i) $\alpha, \beta \in M[J, R[\Omega, R^k]]$, *$\alpha$ and β are lower and upper sample solutions of* (3.2.1) *such that* $\alpha(t,\omega) \le \beta(t,\omega)$;

(ii) *g satisfies the Nagumo condition with respect to α, β, and h_i in* (3.2.11) *satisfies*
$$\int_{\lambda_i(\omega)}^{\infty} \frac{\tau d\tau}{h_i(\tau,\omega)} = \infty$$
where $\lambda_i(\omega) = \max\{|\beta_i(0,\omega) - \alpha_i(1,\omega)|, |\beta_i(1,\omega) - \alpha_i(0,\omega)|\}$ *and* $i \in I_k$. *Then, for any solution $u(t,\omega)$ of* (3.2.1) *with* $\alpha(t,\omega) \le u(t,\omega) \le \beta(t,\omega)$ *on J, there exists a random vector $N(\omega) \ge 0$ depending on α, β and h such that*
$$|u_i'(t,\omega)| \le N_i(\omega) \qquad \text{for } t \in J$$
every $i \in I_k$.

Remark 3.2.4. The validity of Remark 3.2.2 can be justified from the fact that (3.2.9) and (3.2.10) imply

$$|g_i(t,u,v_i,\omega)| \le L_i^1(\omega)|v_i| + L_i^2(\omega) \tag{3.2.14}$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

for $(t, u, v_i, \omega) \in J \times R^k \times R \times \Omega$. Here $h_i(s, \omega) = L_i^1(\omega)s + L_i^2(\omega)$ and it satisfies (3.2.13) for any $\lambda_i \in R[\Omega, R_+]$. Hence for any given random vector $M(\omega) > 0$, we can find a random vector $N(\omega) > 0$, such that

$$|u_i(t, \omega)| \leq M_i(\omega) \qquad \text{and} \qquad |u_i'(t, \omega)| \leq N_i(\omega) \qquad \text{for } t \in J.$$

By using this N, one can define the modified SBVP (3.2.3), and can follow the argument in the proof of Theorem 3.2.1 to conclude the validity of Remark 3.2.2.

A very general comparison theorem can be formulated as follows.

Theorem 3.2.2. *Assume that all the conditions of* Theorem 3.2.1 *are satisfied except* (ii) *and* (iii) *are replaced by*

(ii)′ *$r(t, \omega)$ is the maximal sample solution process of* (3.2.1), *and for every lower solution v of the modified SBVP*

$$u_i'' = g_i^*(t, u, u_i, \omega), \qquad G_\mu(\omega)u(\mu, \omega) = r_\mu(\omega) \qquad (3.2.15)$$

has a sample solution process $\widetilde{u}(t, \omega)$ on J where

$$g_i^*(t, u, u_i', \omega) = g_i(t, p(t, y, \omega), u_i', \omega) + r_i(t, y, \omega),$$

p and r_i are as defined in (3.2.2) *and $i \in I_k$. Then the conclusion of* Theorem 3.2.1 *remains valid.*

Remark 3.2.5. We notice that the Caratheodory type condition on g in (3.2.1) in Theorem 3.2.1 requires the modification in the proof of classical comparison theorems. This modification demands the locally Lipschitz condition on $g(t, u, v, \omega)$ in v for each (t, u, ω).

In the following by employing vector Lyapunov-like functions, we shall develop comparison theorems which have wide applications in the theory of error estimates, stability and solvability analysis of systems of stochastic boundary value problems with random parameters.

Theorem 3.2.3. *Suppose that*

(i) $V \in C^2[J \times R^n, R[\Omega, R^k]]$;

(ii) *for* $(t, x, \omega) \in J \times R^n \times \Omega$,

$$V_f''(t, x, \omega) \geq g(t, V(t, x, \omega), V'(t, x, \omega), \omega)$$

where

$$V_f''(t, x, \omega) = U(t, x, x', \omega) + V_x(t, x, \omega) f(t, x, x', \omega),$$

$$U(t, x, x', \omega) = V_{tt}(t, x, \omega) + 2V_{tx}(t, x, \omega) + V_{xx}(t, x, \omega)(x', x'),$$

$V_{xx}(t, x, \omega)$ *is the bilinear mapping from* $R^n \times R^n$ *into* $R[\Omega, R^k]$, *and*

$$V'(t, x, \omega) = V_t(t, x, \omega) + V_x(t, x, \omega)x';$$

(iii) *Conditions* (i) *and* (iv) *of* Theorem 3.2.1 *are satisfied;*

(iv) $r(t, \omega)$ *is the maximal sample solution process of* (3.2.1) *existing on* J;

(v) *for every lower sample solution process* v, *the modified SBVP* (3.2.15) *has a sample solution process on* J;

(vi) $x(t, \omega)$ *is any sample solution process of* (3.1.1) *existing on* J *such that*

$$G_\mu(\omega) V(\mu, x(\mu, \omega), \omega) \leq r_\mu(\omega), \qquad \text{for } \mu = 0, 1.$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

Then

$$V(t, x(t, \omega), \omega) \leq r(t, \omega) \qquad \textit{for } t \in J.$$

Proof. Let $x(t, \omega)$ be any sample solution process of (3.1.1). We set $m(t, \omega) = V(t, x(t, \omega), \omega)$ and observe from (vi) that

$$G_\mu(\omega) m(\mu, \omega) \leq r_\mu(\omega). \tag{3.2.16}$$

Moreover, using assumptions (i) and (ii), we obtain

$$m''(t, \omega) \geq g(t, m(t, \omega), m'(t, \omega), \omega). \tag{3.2.17}$$

This together with hypotheses (iv), (v), and (3.2.16) satisfies the assumption (ii)$'$ of Theorem 3.2.2. Now an application of Theorem 3.2.2 yields the desired conclusion.

The following comparison theorem is useful for establishing results concerning error analysis and uniqueness of sample solution process (3.2.1).

Theorem 3.2.4. *Assume that all the hypotheses of* Theorem 3.2.3 *are satisfied except* (ii) *and* (vi) *are replaced by*
(ii)$'$ *for* $(t, x, \omega), (t, z, \omega) \in J \times R^n \times \Omega$,

$$V''_{f-\widetilde{f}}(t, x - z, \omega) \geq g(t, V(t, x - z, \omega), V'(t, x - z, \omega), \omega)$$

where

$$\begin{aligned} V''_{f-\widetilde{f}}(t, x - z, \omega) &= U(t, x - z, x' - z', \omega) \\ &\quad + V_x(t, x - z, \omega)(f(t, x, x', \omega) - \widetilde{f}(t, z, z', \omega)), \\ U(t, x - z, x' - z', \omega) &= V_{tt}(t, x - z, \omega) + 2V_{tx}(t, x - z, \omega) \\ &\quad + V_{xx}(t, x - z, \omega)(x' - z', x' - z') \end{aligned}$$

and V_{xx} *is as defined in* Theorem 3.2.3, $\widetilde{f}$ *is either* $\widehat{f}$ *in* (3.1.12) *or* f *in* (3.1.1), *and*

$$V'(t, x-z, \omega) = V_t(t, x-z, \omega) + V_x(t, x-z, \omega)(x' - z');$$

(vi)′ $x(t, \omega)$ *is any sample solution process of* (3.1.1) *existing on* J, *and* $z(t, \omega)$ *is any sample solution process of either* (3.1.2) *or* (3.1.3) *existing on* J *such that*

$$G_\mu(\omega) V(\mu, x(\mu, \omega) - z(\mu, \omega), \omega) \le r_\mu(\omega) \qquad \text{for } \mu = 0, 1,$$

respectively. Then

$$V(t, x(t, \omega) - z(t, \omega), \omega) \le r(t, \omega) \qquad \text{for } t \in J.$$

Proof. The proof of the theorem follows, directly, from the proof of Theorem 3.2.3.

3.3. PROBABILITY DISTRIBUTION METHOD

In this section, we present a method of determining the probability distribution of the solution process of stochastic boundary value problem (3.1.1). The method is based on the Liouville type equation as in Section 2.3. The method does not require the explicit form of the solution process of (3.1.1). Of course, the method is not applicable to very general system of differential equations with random parameters (3.1.1) but it will be limited to a particular class of systems of the following type

$$y'' = h(t, y, y', \eta(t)), \qquad \widehat{B}_\mu y(\mu, \omega) = b_\mu(\omega) \tag{3.3.1}$$

where $\widehat{B}_\mu$ and b_μ are as defined in Section 3.1; $\eta(t) = \xi(t, \nu(\omega))$ is the random coefficient process with $\nu(\omega)$ being a finite-dimensional

random parameter vector defined on a complete probability space $(\Omega, \mathcal{F}, P)$.

We rewrite (3.3.1) as

$$z' = H(t, z) \tag{3.3.2}$$

with

$$Az(0,\omega) + Bz(1,\omega) = c(\omega), \tag{3.3.3}$$

where $z = [y^T, y'^T, \nu^T]^T \in R^N$, $2n < N$, $z(\mu,\omega) = [y^T(\mu,\omega), y'^T(\mu,\omega), \nu^T(\omega)]^T$, for $\mu = 0,1$; $H(t,z) = [y'^T, h^T(t,y,y',\eta(t)), \widetilde{0}^T]^T$; $c(\omega) = [b_0^T(\omega), b_1^T(\omega), \nu^T(\omega)]^T$,

$$A = \begin{bmatrix} \widehat{\alpha}_0 I & -\widehat{\beta}_0 I & \widetilde{0} \\ 0 & 0 & \widetilde{0} \\ 0 & 0 & \widetilde{I} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & \widetilde{0} \\ \widehat{\alpha}_1 I & \widehat{\beta}_1 I & \widetilde{0} \\ 0 & 0 & \widetilde{0} \end{bmatrix},$$

I and $\widetilde{I}$ are identity matrices of dimension $n \times n$ and $(N-2n) \times (N-2n)$,respectively; 0 and $\widetilde{0}$ are zero square matrices of order $n \times n$ and $(N-2n) \times (N-2n)$, respectively. We note that the stochastic boundary value problem (3.3.1) reduces to a first order system of deterministic differential equations (3.3.2) with stochastic boundary conditions. It is assumed that SBVP (3.3.2)–(3.3.3) have a unique sample solution process $z(t,\omega)$ for $t \in [0,1]$. We assume that the components of $b_0(\omega), b_1(\omega)$ and $\nu(\omega)$ have the known joint density function $\widetilde{p}_c(c)$ for $c \in R^N$. We assume that the joint probability density function $p(z,t)$ of the solution process $z(t)$ of (3.3.2)–(3.3.3) exists.

In order to determine $p(z,t)$, we use the Liouville-type theorem (Theorem 2.3.1). For this purpose, we define an auxiliary process $k(t,\omega)$ by

$$k(t,\omega) = Az(0,\omega) + Bz(t,\omega); \tag{3.3.4}$$

then

$$k(1, \omega) = c(\omega) \qquad \text{and} \qquad k(0, \omega) = (A + B)z(0, \omega). \tag{3.3.5}$$

Now we derive the Liouville type equation to determine the joint probability density function for the process $k(t, \omega)$ defined in (3.3.4).

Theorem 3.3.1. *Let $p(k, t)$ be the joint probability density function of $k(t, \omega)$. Then $p(k, t)$ satisfies Liouville equation*

$$\frac{\partial p}{\partial t} + \sum_{\ell, m=1}^{2n} b_{\ell m} \frac{\partial (pF_m(t, x, \xi(t, g)))}{\partial x_m} = 0 \tag{3.3.6}$$

where $x = [y^T, y'^T]^T$, $y, y' \in R^n$, $g \in R^{N-2n}$ and $B = (b_{ij})_{N \times N}$; $F(t, x, \xi(t, g)) = [y'^T, h^T(t, y, y', \xi(t, g))]^T$ and h and ξ are as described in (3.3.1).

Proof. As in Theorem 2.3.1, we denote the joint characteristic function of $k(t, \omega)$ by $\phi(u, t)$, where

$$\phi(u, t) = E\left[\exp\left[i\left(\sum_{j=1}^{N} u_j k_j(t, \omega)\right)\right]\right], \qquad i = \sqrt{-1}. \tag{3.3.7}$$

By following the proof of Theorem 2.3.1, we arrive at

$$\begin{aligned}
\frac{\partial \phi}{\partial t} &= E\left[\frac{\partial}{\partial t}\exp\left[i\left(\sum_{j=1}^{N} u_j k_j(t, \omega)\right)\right]\right] \\
&= E\left[i\sum_{\ell=1}^{N} u_\ell \frac{\partial k_\ell(t, \omega)}{\partial t}\exp\left[i\left(\sum_{j=1}^{N} u_j k_j(t, \omega)\right)\right]\right] \\
&= E\left[i\sum_{\ell=1}^{N} u_\ell\left(\sum_{m=1}^{N} b_{\ell m} H_m(t, z(t, \omega))\right)\exp\left[i\left(\sum_{j=1}^{N} u_j k_j(t, \omega)\right)\right]\right] \\
&= i\sum_{\ell=1}^{N} u_\ell \sum_{m=1}^{N} b_{\ell m} E\left[H_m(t, z(t, \omega))\exp\left[i\left(\sum_{j=1}^{N} u_j k_j(t, \omega)\right)\right]\right].
\end{aligned} \tag{3.3.8}$$

To replace $z(t,\omega)$ in terms of $k(t,\omega)$, we solve the following stochastic boundary value problem

$$z' = H(s,z), \qquad s \in [0,1] \tag{3.3.9}$$

with

$$Az(0,\omega) + Bz(t,\omega) = k(t,\omega) \tag{3.3.10}$$

whose solution we denote by $z = z^*(s, k(t,\omega))$. Observing that

$$z^*(t, k(t,\omega)) = z(t,\omega), \tag{3.3.11}$$

we can set $s = t$ in z^* and substitute into (3.3.8). This gives

$$\frac{\partial \phi}{\partial t} = i\sum_{\ell=1}^{N} u_\ell \sum_{m=1}^{N} b_{\ell m} E\left[H_m(t, z(t, k(t,\omega))) \exp\left[i\left(\sum_{j=1}^{N} u_j k_j(t,\omega)\right)\right]\right].$$

This together with properties of the Fourier transform, one obtains,

$$\frac{\partial \phi}{\partial t} = -\int_{-\infty}^{\infty} \exp\left[i\left(\sum_{j=1}^{N} u_j k_j\right)\right] \sum_{\ell,m=1}^{N} b_{\ell m} \frac{\partial}{\partial z_m}\left(H_m(k,t)p(k,t)\right)dk. \tag{3.3.12}$$

On the other hand upon differentiating (3.3.7) with respect to t, we get

$$\frac{\partial \phi}{\partial t} = \int_{-\infty}^{\infty} \exp\left[i\left(\sum_{j=1}^{N} u_j k_j\right)\right] \frac{\partial}{\partial t} p(k,t)dk.$$

From the above equation and (3.3.12), we get

$$\frac{\partial}{\partial t} p(k,t) + \sum_{\ell,m=1}^{N} b_{\ell m} \frac{\partial (p(k,t) H_m(k,t))}{\partial k_m} = 0. \tag{3.3.13}$$

From this together with the definitions of H and B yields

$$\frac{\partial p(k,t)}{\partial t} + \sum_{\ell,m=1}^{2n} b_{\ell m} \frac{\partial(p(k,t)F_m(t,k_1,k_2,\ldots,k_{2n},\xi(t,k_{2n+1},\ldots,k_N)))}{\partial k_m} = 0.$$

This shows that $p(k,t)$ satisfies (3.3.6). The proof of the theorem is complete.

A formal explicit solution of (3.3.6) can be found by means of its associated Lagrange system. For this purpose, (3.3.6) can be rewritten as

$$\frac{\partial p}{\partial t} + \left(\sum_{\ell,m=1}^{2n} b_{\ell m}\frac{\partial F_m}{\partial k_m}\right) p + \sum_{\ell,m=1}^{2n} b_{\ell m} F_m \frac{\partial p}{\partial k_m} = 0. \tag{3.3.14}$$

From the particular structure of SBVP (3.3.2)–(3.3.3), (3.3.14) can be further simplified to

$$\frac{\partial p}{\partial t} + \left(\sum_{\ell=1}^{2n}\sum_{m=n+1}^{2n} b_{\ell m}\frac{\partial F_m}{\partial k_m}\right) p + \sum_{\ell,m=1}^{2n} b_{\ell m} F_m \frac{\partial p}{\partial k_m} = 0. \tag{3.3.15}$$

The Lagrange system (3.3.14) in the context of (3.3.15) is described by

$$\begin{aligned}\frac{dt}{1} &= \frac{dk_1}{\sum_{\ell=1}^{2n} b_{\ell 1}F_1} = \frac{dk_2}{\sum_{\ell=1}^{2n} b_{\ell 2}F_2} \\ &= \cdots = \frac{dk_{2n}}{\sum_{\ell=1}^{2n} b_{\ell 2n}F_{2n}} = -\frac{dp}{(\nabla_k \cdot F)\,p},\end{aligned} \tag{3.3.16}$$

where $\nabla_k = \left(\sum_{\ell=1}^{2n} b_{\ell 1}\frac{\partial}{\partial k_1}, \sum_{\ell=1}^{2n} b_{\ell 2}\frac{\partial}{\partial k_2}, \ldots, \sum_{\ell=1}^{2n} b_{\ell 2n}\frac{\partial}{\partial k_{2n}}\right)$. From the definitions of B and F (3.3.16) can be further rewritten as

$$\begin{aligned}\frac{dt}{1} &= \frac{dk_1}{\widehat{\alpha}_1 k_{n+1}} = \frac{dk_2}{\widehat{\alpha}_1 k_{n+2}} = \cdots = \frac{dk_n}{\widehat{\alpha}_1 k_{2n}} = \frac{dk_{n+1}}{\widehat{\beta}_1 h_1} \\ &= \frac{dk_{n+2}}{\widehat{\beta}_1 h_2} = \cdots = \frac{dk_{2n}}{\widehat{\beta}_1 h_n} = -\frac{dp}{\left(\widehat{\alpha}_1 + \widehat{\beta}_1\right)(\nabla_k \cdot h)\,p}\end{aligned} \tag{3.3.17}$$

where $\nabla_k = \left(\frac{\partial}{\partial k_{n+1}}, \frac{\partial}{\partial k_{n+2}}, \ldots, \frac{\partial}{\partial k_{2n}}\right)$. System (3.3.16) is equivalent to the following first-order system of deterministic differential equations

$$\begin{cases} \dfrac{dk}{dt} = F^*(t,k), & k(1) = c \\ \dfrac{dp}{dt} = -\nabla_k \cdot F(t,k)p, & p(k,1) = \widetilde{p}_c(c) \end{cases} \tag{3.3.18}$$

where $x = [k_1, k_2, \ldots, k_{2n}]^T \in R^{2n}$ and ∇_k as defined in (3.3.16);

$$F^*(t,k) = \left[\sum_{\ell=1}^{2n} b_{\ell 1}F_1, \sum_{\ell=1}^{2n} b_{\ell 2}F_2, \ldots, \sum_{\ell=1}^{2n} b_{\ell 2n}F_{2n}, 0, \ldots, 0\right]^T.$$

A sufficient condition of the solution of system (3.3.18) is given by the following theorem.

Theorem 3.3.2. *Assume that* $\frac{\partial F}{\partial k}(t,k)$ *exists and satisfies Caratheodory-type condition. Furthermore, the solution* $k(t) = k(t,1,k(1))$ *of the following system*

$$\frac{dk}{dt} = F^*(t,k), \qquad k(1) = c$$

has the inverse transform $k(1) = \tau(k,t,1)$ *for* $0 \le t \le 1$. *Then*

$$p(k,t) = \widetilde{p}(c)\exp\left[\int_t^1 \nabla_k \cdot F^*(s,k(s))ds\right]\Bigg|_{k(1)=\tau(k,t,1)}. \tag{3.3.19}$$

Proof. The proof of the theorem can be formulated analogous to the proof of Theorem 2.3.2.

The following result provides the tool to find the probability density of solution process of (3.3.1).

Theorem 3.3.3. *Let the hypotheses of* Theorem 3.3.2 *be satisfied. Further assume that* $(A+B)$ *in* (3.3.5) *invertible. Then the joint probability density* $p(z,t)$ *of the solution process of* (3.3.2)–(3.3.3) *is given by*

$$p(z,t) = p(z_0, 0) \exp\left[-\int_0^t \nabla_z \cdot F(s, z(s))\right]\Bigg|_{z_0=\mu(z,t)}, \qquad (3.3.20)$$

where $p(z_0, 0)$ *is the probability density function* $z(0,\omega)$.

Proof. From (3.3.5), we recall

$$k(0,\omega) = (A+B)z(0,\omega).$$

Thus the probability density of $z(0,\omega)$ is given by

$$p(z_0, 0) = p((A+B)z_0, 0)\det((A+B)^{-1}). \qquad (3.3.21)$$

The joint probability density function of the solution function of SBVP (3.3.2)–(3.3.3) is the same as the joint probability density function the following initial value problem with random parameters

$$z' = H(t,z), \qquad z(0,\omega) = z_0(\omega). \qquad (3.3.22)$$

From the assumption of the theorem and the application of Theorems 2.3.1 and 2.3.2 to the SIVP (3.3.22), the conclusion of the theorem follows from (2.3.12), immediately. This completes the proof of the theorem.

Remark 3.3.1. It is clear that the marginal probability density function of the SBVP (3.3.2)–(3.3.3) can be determined from (3.3.20) by integration, that is

$$p(y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(z,t)dy'dg = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(y,y',g,t)dy'dg. \qquad (3.3.23)$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Example 3.3.1. We consider the linear two-point stochastic boundary value problem

$$y'' = Q_0(\nu_0(\omega), t)y + Q_1(\nu_1(\omega), t)y' + q_0(t) \tag{3.3.24}$$

with

$$\bar{A}_\mu y(\mu, \omega) + \bar{B}_\mu y'(\mu, \omega) = b_\mu(\omega), \qquad \mu = 0, 1 \tag{3.3.25}$$

where $y \in R^n$, $q_0(t)$ is deterministic function; Q_0 and Q_1 are $n \times n$ matrix functions with random parameters ν_0 and ν_1; $\bar{A}_\mu$ and $\bar{B}_\mu$ are $n \times n$ constant matrices and b_μ is an n-dimensional random vector; let $\widetilde{p}(c)$ be the joint probability density function of the components of $b_0(\omega)$, $b_1(\omega)$, $\nu_0(\omega)$ and $\nu_1(\omega)$.

By setting $z = [y^T, y'^T, \nu_0^T, \nu_1^T]^T \in R^N$, $2n < N$, $z(\mu, \omega) = [y^T(\mu, \omega), y'^T(\mu, \omega), \nu_0^T(\omega), \nu_1^T(\omega)]^T$ for $\mu = 0, 1$, and $c(\omega) = [b_0^T(\omega), b_1^T(\omega), \nu_0^T(\omega), \nu_1^T(\omega)]^T$, SBVP (3.3.24)–(3.3.25) can be written as

$$\frac{dz}{dt} = Q(\nu(\omega), t)z + q(t), \tag{3.3.26}$$

with

$$Az(0, \omega) + Bz(1, 0) = c(\omega), \tag{3.3.27}$$

where $q(t) = [0^T, q_0^T(t), \widetilde{0}^T]^T$; $\nu(\omega) = [\nu_0^T(\omega), \nu_1^T(\omega)]^T$

$$Q(\nu(\omega), t) = \begin{bmatrix} 0 & I & \widetilde{0} \\ Q_0(\nu_0(\omega), t) & Q_1(\nu_1(\omega), t) & \widetilde{0} \\ 0 & 0 & \widetilde{0} \end{bmatrix}$$

$$A = \begin{bmatrix} \bar{A}_0 & \bar{B}_0 & \widetilde{0} \\ 0 & 0 & \widetilde{0} \\ 0 & 0 & \widetilde{I} \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 0 & \widetilde{0} \\ \bar{A}_1 & \bar{B}_1 & \widetilde{0} \\ 0 & 0 & \widetilde{0} \end{bmatrix} = (b_{ij})_{N \times N}.$$

0, $\widetilde{0}$, I and $\widetilde{I}$ are as defined in (3.3.3).

Let $z(t,\omega)$ be the solution process of (3.3.26)–(3.3.27). The probability density for the process $k(t,\omega)$ in (3.3.4) relative SBVP (3.3.26)–(3.3.27) satisfies (3.3.6) where F_m is m-th component of $Q(g,t)z$ where $1 \le m \le 2n$ and $g \in R^{N-2n}$. The verification of (3.3.11) relative to the linear system is given in the following lemma.

Lemma 3.3.1. *Assume that Q and q are continuous on $[0,1]$. Let $\phi(t,g)$ be the fundamental matrix solution process of*

$$u' = Q(g,t)u \tag{3.3.28}$$

with $\phi(0) = I$. Further assume that $A + B\phi(t,g)$ is non-singular for $t \in [0,1]$. Then

(i) *For $1 \le j \le N$,*

$$u^j(t,g) = \phi(t,g)[A + B\phi(1,g)]^{-1}e^j \tag{3.3.29}$$

is the solution process of following SBVP

$$u' = Q(g,t)u \tag{3.3.30}$$

with

$$Au(0,g) + Bu(1,g) = e^j, \tag{3.3.31}$$

where e^j's form a orthonormal basis for R^N, moreover

$$\widehat{\phi}(t,g) = \phi(t,g)[A + B\phi(1,g)]^{-1} \tag{3.3.32}$$

is a fundamental matrix solution of (3.3.28);

(ii) *the solution process $\widehat{z}(t,g)$ of the following boundary value problem*

$$\frac{dz}{dt} = Q(g,t)z + q(t) \tag{3.3.33}$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

with

$$Az(0,g) + Bz(1,g) = 0 \tag{3.3.34}$$

is given by

$$\widehat{z}(t,g) = -\phi(t,g)[A + B\phi(1,g)]^{-1}B\int_0^1 \phi(1,s,g)q(s)ds$$
$$+ \int_0^t \phi(t,s,g)q(s)ds, \qquad t \in [0,1]; \tag{3.3.35}$$

(iii) *the solution process* $z(t,\omega)$ *of* (3.3.26)–(3.3.27) *is given by*

$$z(t,\omega) = \widehat{\phi}(t,g)c(\omega) + \widehat{z}(t,g); \tag{3.3.36}$$

(iv)
$$z^*(t,k(t,\omega)) = z(t,\omega), \tag{3.3.37}$$

where for $s \in [0,t]$, $z^*(s,k(t,\omega))$ *is a solution process of the following SBVP*

$$z' = Q(g,s)z \tag{3.3.38}$$

with

$$Az(0,\omega) + Bz(t,\omega) = k(t,\omega), \tag{3.3.39}$$

$z(t,\omega)$ *is the solution process of* (3.3.26)–(3.3.27).

Proof. The general solution (3.3.28) is given by

$$u(t,g) = \phi(t,g)a,$$

where a is an arbitrary N-dimensional constant vector. By applying the boundary conditions (3.3.31), we obtain

$$[A + B\phi(1,g)]a = e^j.$$

This together with the nonsingularity assumption of $A + B\phi(t,g)$ on $[0,1]$, yields

$$u^j(t,g) = \phi(t,g)[A + B\phi(1,g)]^{-1}e^j, \qquad j = 1,2,\ldots,N. \tag{3.3.40}$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

This establishes the relation (3.3.29). The nonsingularity of $A + B\phi(t,g)$ on $[0,1]$ also guarantees the linear independence of $u^1(t,g)$, $u^2(t,g), \ldots, u^N(t,g)$ on $[0,1]$. Thus the matrix $\widehat{\phi}(t,g) = \phi(t,g)[A + B\phi(1,g)]^{-1}$ is a fundamental matrix solution of (3.3.28). Its j-th column is $u^j(t,g)$. This completes the proof of (i).

To prove (ii), we consider the general solution of (3.3.33)–(3.3.34) in terms of $\phi(t,g)$

$$\widehat{z}(t,g) = \phi(t,g)a + \int_0^t \phi(t,s,g)q(s)ds,$$

where a is an arbitrary constant vector. This together with the boundary conditions (3.3.34) yields

$$[A + B\phi(1,g)]a = -B\int_0^1 \phi(1,s,g)q(s)ds,$$

and hence $\widehat{z}(t,g)$ is also uniquely determined:

$$\begin{aligned}\widehat{z}(t,g) = &- \phi(t,g)[A + B\phi(1,g)]^{-1}B\int_0^1 \phi(1,s,g)q(s)ds \\ &+ \int_0^t \phi(t,s,g)q(s)ds.\end{aligned}$$

This proves the part (ii). The proof of (iii) follows from (i) and (ii). Details are left to the reader. From (3.3.36), we note that $z(t,\omega)$ is the solution of the initial value problem

$$\frac{dz}{dt} = Q(g,t) + q(t), \qquad z(0,\omega) = z_0 \tag{3.3.41}$$

where

$$\begin{aligned}z_0 &= \widehat{\phi}(0,g)c(\omega) + \widehat{z}(0,g) \\ &= [A + B\phi(1,g)]^{-1}\left[c(\omega) - B\int_0^1 \phi(1,s,g)q(s)ds\right].\end{aligned} \tag{3.3.42}$$

In fact, from (3.3.32), (3.3.35) and (3.3.42), (3.3.36) can be rewritten as

$$
\begin{aligned}
z(t,\omega) &= \phi(t,g)[A+B\phi(1,g)]^{-1}c(\omega) \\
&\quad - \phi(t,g)[A+B\phi(1,g)]^{-1}B\int_0^1 \phi(1,s,g)q(s)ds \\
&\quad + \int_0^t \phi(t,s,g)q(s)ds \\
&= \phi(t,g)\left[(A+B\phi(1,g))^{-1}\left(c(\omega)-B\int_0^1 \phi(1,s,g)q(s)ds\right)\right] \\
&\quad + \int_0^t \phi(t,s,g)q(s)ds \\
&= \phi(t,g)z_0 + \int_0^t \phi(t,s,g)q(s)ds. \qquad (3.3.43)
\end{aligned}
$$

The right-hand side expression in (3.3.43) is the solution process of (3.3.41)–(3.3.42). In addition, the solution of the initial value problem (3.3.41)–(3.3.42) has the representation as described in (3.3.36). In short, the solution SBVP (3.3.26)–(3.3.27) is the same as the solution of the initial value problem (3.3.41)–(3.3.42).

To prove (iv), we apply (iii) to the solution process of SBVP (3.3.38)–(3.3.39). Let $z^*(s,k(t,\omega))$ be the solution process of (3.3.38)–(3.3.39) for $s\in[0,t]$. From the part (ii) and (3.3.43), we have

$$
z^*(s,k(t,\omega)) = \phi(s,g)z^* + \int_0^s \phi(s,u,g)q(u)du, \qquad s\in[0,t], \tag{3.3.44}
$$

where

$$
z^* = [A+B\phi(t,g)]^{-1}\left[k(t,\omega) - B\int_0^t \phi(t,s,g)q(s)ds\right]. \tag{3.3.45}
$$

On the other hand from the definition of $k(t,\omega)$ in (3.3.39) and the

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

right-hand side expression for $z(t,\omega)$ in (3.3.43), we have

$$k(t,\omega) = Az(0,\omega) + B\left(\phi(t,g)z_0 + \int_0^t \phi(t,s,g)q(s)ds\right)$$
$$= (A + B\phi(t,g))z_0 + B\int_0^t \phi(t,s,g)q(s)ds.$$

From this, (3.3.45) becomes

$$z^* = (A + B\phi(t,g))^{-1}\Big[(A + B\phi(t,g))z_0 + B\int_0^t \phi(t,s,g)q(s)ds$$
$$- B\int_0^t \phi(t,s,g)q(s)ds\Big] = z_0.$$

This together with (3.3.43) and (3.3.44) yields

$$z^*(s,k(t,\omega)) =$$
$$\phi(s,g)z_0(\omega) + \int_0^s \phi(s,u,g)q(u)du = z(s,\omega) \quad \text{for } s \in [0,t].$$

From this the proof of (3.3.37) follows, immediately. This completes the proof of the lemma.

From the above discussion and the applications of Theorems 3.3.2 and 3.3.3, the joint probability density of the solution process of (3.3.26)–(3.3.27) can be computed from (3.3.20). In fact, the expression of $p(z,t)$ in (3.3.20) relative to the solution process of (3.3.26)–(3.3.27) becomes

$$p(z,t) = p(z_0,0)\exp\left[-\sum_{j=1}^n \int_0^t q_{jj}(g,s)ds\right]$$
$$= p(\mu(z,t),0)\exp\left[-\sum_{j=1}^n \int_0^t q_{jj}(g,s)ds\right].$$

From this (3.3.19), (3.3.21), and part (ii) of Lemma 3.3.1, we have

$$p(z,t) = \widetilde{p}(c)\exp\left[\sum_{j=1}^n \int_0^1 q_{jj}(g,s)ds\right]\exp\left[-\sum_{j=1}^n \int_0^t q_{jj}(g,s)ds\right]$$
$$= \widetilde{p}(\tau(z,t,1))\exp\left[\sum_{j=1}^n \int_t^1 q_{jj}(g,s)ds\right]\det((A+B)^{-1})$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

where $\tau(z,t,1) = c = \widehat{\phi}^{-1}(t,g)[z - \widehat{z}(t,g)]$. The rest of the discussion is left to the reader as an exercise.

Remark 3.3.2. We remark that if Q_0 and Q_1 are deterministic continuous functions, then the probability density of the solution process of (3.3.24)–(3.3.25) can be computed from the above expression as well as from (3.3.36). This is because of the fact that the fundamental matrix solution $\widehat{\phi}$ in (3.3.32) with $z = x \in R^{2n}$. In fact,

$$c(\omega) = [A + B\phi(1)]\phi^{-1}(t)[x(t,\omega) - \widehat{x}(t)],$$

where

$$c(\omega) = [b_0^T(\omega), B_1^T(\omega)]^T, \quad A = \begin{bmatrix} \bar{A} & \bar{B} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ \bar{A}_1 & \bar{B}_1 \end{bmatrix}.$$

Hence for $1 \le i \le 2n$

$$c_i(\omega) = \sum_{j=1}^{2n} \widehat{\phi}_{ij}^{-1}(t)[x_j(t,\omega) - \widehat{x}_j(t)] = L_i(x_1(t,\omega), \ldots, x_{2n}(t,\omega)), \tag{3.3.46}$$

where $\widehat{\phi}_{ij}^{-1}$ is the element in the i-th row and j-th column of $\widehat{\phi}^{-1}(t)$. Equations (3.3.46) provide an explicit (linear) relation between $c(\omega)$ and $x(t,\omega)$. This linear relation relationship enables us to relate their respective probability densities $\widetilde{p}(c)$ and $p(x,t)$. Thus

$$p(x,t) = \widetilde{p}(c)\det(\widehat{\phi}^{-1}(t)) = \widetilde{p}(L(x,t))\det(\widehat{\phi}^{-1}(t)), \tag{3.3.47}$$

where

$$c_i = L_i(x,t) = \sum_{j=1}^{2n} \widehat{\phi}_{ij}^{-1}(t)[x - \widehat{x}(t)], \qquad i = 1, 2, \ldots, 2n. \tag{3.3.48}$$

In the following, we provide some examples to illustrate the results.

Example 3.3.2. Suppose that

$$y'' + 3y' + 2y = 0, \qquad y(0,\omega) = b_0(\omega), \qquad y'(1,\omega) = b_1(\omega) \quad (3.3.49)$$

where $b_0(\omega)$ and $b_1(\omega)$ are independent normal random variables. Thus

$$\widetilde{p}(c) = \frac{1}{2\pi}\exp[-(c_1^2 + c_2^2)/2]. \qquad (3.3.50)$$

Set $y = x_1$, and $x_2 = y'$, then (3.3.49) becomes

$$\frac{dx}{dt} = Qx, \qquad (3.3.51)$$

where

$$Q = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case, we have

$$\phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix},$$

$$A + B\phi(1) = \begin{bmatrix} 1 & 0 \\ \frac{-2e+2}{e^2} & \frac{-e+2}{e^2} \end{bmatrix}$$

and

$$\phi^{-1}(t) = \begin{bmatrix} -e^{2t} + 2e^t & -e^{2t} + e^t \\ 2e^{2t} - 2e^t & 2e^{2t} - e^t \end{bmatrix}.$$

In this case, (3.3.36) becomes

$$x(t,\omega) = \widehat{\phi}(t)c(\omega)$$

and hence

$$\begin{aligned} c(\omega) &= \widehat{\phi}^{-1}(t)x(t,\omega) = [A + B\phi(1)]\phi^{-1}(t)x(t,\omega) \\ &= \begin{bmatrix} 1 & 0 \\ 2e^{-2}(1-e) & e^{-2}(2-e) \end{bmatrix} \begin{bmatrix} -e^{2t} + 2e^t & -e^{2t} + e^t \\ 2e^{2t} - 2e^t & 2e^{2t} - e^t \end{bmatrix} x(t,\omega) \\ &= \begin{bmatrix} 2e^t - e^{2t} & e^t - e^{2t} \\ 2e^{-2}(e^{2t} - ee^t) & e^{-2}(2e^{2t} - ee^t) \end{bmatrix} x(t,\omega) \end{aligned}$$

which implies that

$$\begin{bmatrix} b_0(\omega) = (2e^t - e^{2t})x_1(t,\omega) + (e^t - e^{2t})x_2(t,\omega) \\ b_1(\omega) = 2e^{-2}(e^{2t} - ee^t)x_1(t,\omega) + e^{-2}(2e^{2t} - ee^t)x_2(t,\omega) \end{bmatrix}. \tag{3.3.52}$$

Here

$$c = \begin{bmatrix} L_1(x,t) \\ L_2(x,t) \end{bmatrix} = \begin{bmatrix} (2e^t - e^{2t})x_1 + (e^t - e^{2t})x_2 \\ 2e^{-2}(e^{2t} - ee^t)x_1 + e^{-2}(2e^{2t} - ee^t)x_2 \end{bmatrix}. \tag{3.3.53}$$

From (3.3.50), (3.3.52), (3.3.53), and using formula (3.3.47), we obtain

$$\begin{aligned} p(x,t) &= \widetilde{p}(c)\det(\widehat{\Phi}^{-1}(t)) = \widetilde{p}(L(x,t))\det(\widehat{\Phi}^{-1}(t)) \\ &= \frac{\det(\widehat{\phi}^{-1}(t))}{2\pi}\exp\Big[-\Big(\big[(2e^t - e^{2t})x_1 + (e^t - e^{2t})x_2\big]^2 \\ &\qquad + \big[2e^{-2}(e^{2t} - ee^t)x_1 + e^{-2}(2e^{2t} - ee^t)x_2\big]^2\Big)/2\Big] \\ &= \frac{(2-e)}{2\pi e^2}\exp\Big[\Big(6t - [e^t(2x_1 + x_2) - e^{2t}(x_1 + x_2)]^2 \\ &\qquad + [2e^{-2+2t}(x_1 + x_2) + e^{-2+t}(x_2 - 2x_1)]^2\Big)/2\Big]. \end{aligned} \tag{3.3.54}$$

Finally, the probability density function of solution process $y(t,\omega)$ of (3.3.49) is given by

$$p(y,t) = \int_{-\infty}^{\infty} p(x_1, x_2, t)\,dx_2 \tag{3.3.55}$$

where $p(x,t)$ is as defined in (3.3.54).

Remark 3.3.3. A remark similar to Remark 2.3.3 with regard to boundary value problems with random parameters can be formulated, analogously.

3.4. SOLVABILITY AND UNIQUENESS ANALYSIS

The general comparison theorems that are developed in Section 3.2, will be utilized to prove the existence and uniqueness results

for sample solution processes of stochastic boundary value problem (3.1.1).

The following result provides a set of sufficient conditions for the existence of sample solutions process.

Theorem 3.4.1. *Assume that*

(A_0) *the hypothesis of* Lemma 3.1.1 *hold;*

(A_1) $V \in C^2[J \times R^n, R[\Omega, R^n]]$ *and* $\sum_{i=1}^{k} V_i(t, x, \omega) \to \infty$ *as* $\|x\| \to \infty$ *uniformly in* $t \in J$*;*

(A_2) $g \in M[J \times R^k \times R^k, R[\Omega, R^k]]$, $g(t, u, v, \omega)$ *satisfies hypothesis* (H_1), *the quasi-monotone non-increasing property in* u *for each* $(t, v, \omega) \in J \times R^k \times \Omega$ *and the locally Lipschitz condition in* v*;*

(A_3) *for* $0 \leq \eta \leq 1$

$$V''_{\eta\eta f}(t, x, \omega) \geq g(t, V(t, x, \omega), V'(t, x, \omega), \omega),$$

where

$$V''_{\eta\eta f}(t, x, \omega) = U(t, x, x', \omega) + \eta(x, \omega)\eta(x', \omega)V_x(t, x, \omega)f(t, x, x', \omega),$$

$$U(t, x, x', \omega) = V_{tt}(t, x, \omega) + 2V_{tx}(t, x, \omega) + V_{xx}(t, x, \omega)(x', x'),$$

$$V'(t, x, \omega) = V_t(t, x, \omega) + V_x(t, x, \omega)x',$$

and $V_{xx}(t, x, \omega)$ *is the bilinear map from* $R^n \times R^n$ *into* $R[\Omega, R^k]$*;*

(A_4) *the boundary conditions in* (3.1.1) *satisfy*

$$G_\mu(\omega)V(\mu, x(\mu, \omega), \omega) \leq r_\mu(\omega) \qquad \text{for } \mu = 0, 1;$$

(A_5) $r(t, \omega)$ *is the maximal sample solution process of* (3.2.1)*;*

(A_6) *for every lower sample solution process* m, *the modified SBVP* (3.2.15) *has a sample solution process on* J*;*

(A_7) *for* $x \in C^2[J, R[\Omega, R^n]]$ *such that*

$$\|x''\| \leq h(\|x'\|, \omega), \qquad \text{for } \|x\| \leq M(\omega)$$

where h *is a positive non-decreasing sample continuous process such that*

$$\lim_{s \to \infty} \frac{s^2}{h(s, \omega)} = \infty$$

and M *is a positive random variable. Then there exist a sample solution process of* (3.1.1) *on* J.

Proof. Let $\eta(x, \omega)$ be a sample continuous process with values in [0,1] such that $\eta(x, \omega) = 1$ for$\|x\| \leq M(\omega)$ and $\eta(x, \omega) = 0$ for $\|x\| \geq M(\omega) + 1$. The positive random variable M is to be specified later. The modified function of f in (3.1.1) is defined by

$$\widetilde{f}(t, x, y, \omega) = \eta(x, \omega)\eta(y, \omega) f(t, x, y, \omega). \tag{3.4.1}$$

It can be easily seen that $\widetilde{f}$ satisfies (H_1) and (3.1.7). Thus by Theorem 3.1.2, the SBVP

$$y'' = \widetilde{f}(t, y, y', \omega), \qquad B_\mu(\omega) y(\mu, \omega) = b_\mu(\omega) \tag{3.4.2}$$

has a sample solution process $y(t, \omega)$. Setting $m(t, \omega) = V(t, y(t, \omega), \omega)$ and using the assumptions (A_3) and (A_4) we get

$$m''(t, \omega) \geq g(t, m(t, \omega), m'(t, \omega), \omega), \qquad G_\mu(\omega) m(\mu, \omega) \leq r_\mu(\omega).$$

This together with (A_2), (A_5), (A_6) and the application of Theorem 3.2.2 yields

$$m(t, \omega) \leq r(t, \omega) \qquad \text{on } J,$$

which implies that

$$\sum_{i=1}^{k} V_i(t, y(t,\omega), \omega) \leq \sum_{i=1}^{k} r_i(t,\omega) \qquad \text{on } J,$$

where $r(t,\omega)$ is the maximal sample solution process of (3.2.1). Now the assumption (A_0) implies that there exists a positive random variable $M_1(\omega)$ such that $\|y(t,\omega)\| \leq M_1(\omega)$ on J. Choosing this $M_1(\omega)$ in Lemma 3.2.1, one obtains $\|y'(t,\omega)\| \leq M_2(\omega)$ on J. The positive random variable $M(\omega)$ defined by $M(\omega) = \max\{M_1(\omega), M_2(\omega)\}$ is used in the definition of η. In view of the definition of $\widetilde{f}$, it follows that $f \equiv \widetilde{f}$ and so $y(t,\omega)$ is actually a sample solution process of (3.1.1). The proof is thus complete.

For the discussion of the uniqueness result, let us consider simple boundary conditions in (3.1.1) and (3.2.1), namely,

(H_2') $\alpha_0 = 1 = \beta_1$, $\alpha_1 = 0 = \beta_0$; $\widehat{\alpha}_0 = 1 = \widehat{\beta}_1$, $\widehat{\alpha}_1 = 0 = \widehat{\beta}_0$ and $\gamma_0 = 1 = \nu_1$, $\gamma_1 = 0 = \nu_0$.

The following result provides the uniqueness of sample solution process of (3.1.1) in the context of (H_2').

Theorem 3.4.2. *Assume that*

(i) $V \in C^2[J \times R^n, R[\Omega, R_+^k]]$, $V(t,0,\omega) \equiv 0$ *iff* $x = 0$ *and*

$$V_f''(t, x-y, \omega) \geq g(t, V(t, x-y, \omega), V'(t, x-y, \omega), \omega)$$

V'' and V' are having similar expressions as before;

(ii) *g satisfies the assumptions (A_2), (A_5) and (A_6) of* Theorem 3.4.1;

(iii) *$u(t,\omega) \equiv 0$ is the only sample solution process of SBVP* (3.2.1) *in the context of (H_2') with $r_0 = r_1 = 0$.*

Then the SBVP (3.1.1) *in the context of (H_2') has at most one sample solution process.*

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Proof. Let $x(t,\omega)$ and $y(t,\omega)$ be two sample solution processes of the SBVP (3.1.1) in the context of (H_2'). Set $m(t,\omega) = V(t, x(t,\omega) - y(t,\omega), \omega)$ and note $m(0,\omega) = 0$, $m(1,\omega) = 0$. Hence by comparison Theorem 3.2.2, we have $m(t,\omega) \leq r(t,\omega)$ on J, where $r(t,\omega)$ is the maximal sample solution process of (3.2.1) with $u(0,\omega) = u(1,\omega) = 0$. By assumption (iii), $r(t,\omega) \equiv 0$ and this implies $x(t,\omega) \equiv y(t,\omega)$ on J proving the theorem.

3.5. STABILITY ANALYSIS

In this section, we first formulate stochastic stability concepts, and then present some sufficient conditions for the stability of the trivial solution process of (3.1.1). We note that (3.1.1) and (3.2.1) in the context of (H_2') have unique sample trivial solution processes, respectively, whenever $f(t,0,0,\omega) \equiv 0$, $g(t,0,0,\omega) \equiv 0$ with $b_\mu = 0$ and $r_\mu = 0$ for $\mu = 0, 1$.

Definition 3.5.1. The trivial solution process of (3.1.1) in the context of (H_2') is said to be

(SMB) *stable in the p-th moment*, if for each $\epsilon > 0$ and $p \geq 1$, there exist positive real numbers δ_0 and δ_1 such that for $\mu = 0, 1$, the inequalities $\|b_\mu\|_p \leq \delta_\mu$ implies

$$\|y(t,\omega)\|_p < \epsilon, \qquad t \in J$$

where

$$\|y(t,\omega)\|_p = \left(\int_\Omega \|y(t,\omega)\|^p P(d\omega) \right)^{1/p}.$$

Remark 3.5.1. We note that depending on the mode of convergence in the probabilistic analysis, one can formulate other definitions of stochastic stability and boundedness. See Ladde and Lakshmikantham [67].

The use of comparison method to study the stability analysis of (3.1.1) requires the knowledge about the stability of comparison system (3.2.1). For example the stability concepts in Definition 3.5.1 relative to (3.2.1) in the context of (H_2') will be defined as follows:

Definition 3.5.2. The trivial solution process $u \equiv 0$ of (3.2.1) in the context of (H_2') is said to be

(SMB*) *stable in the mean*, if for $\epsilon > 0$, there exist positive numbers δ_μ^* for $\mu = 0, 1$ such that $\sum_{i=1}^k E[r_{\mu i}] \leq \delta_\mu^*$ implies

$$\sum_{i=1}^k E[u_i(t,\omega)] < \epsilon \qquad \text{for } t \in J.$$

We shall present a stability result that establishes the p^{th} moment stability of the trivial solution process of (3.1.1) in the context of boundary conditions (H_2').

Theorem 3.5.1. *Let the hypotheses of* Theorem 3.2.3 *be satisfied. Further assume that* $f(t,0,0,\omega) \equiv 0$ *and* $g(t,0,0,\omega) = 0$ *with probability one, and for* $(t,x) \in R_+ \times R^n$

$$b(\|x\|^p) \leq \sum_{i=1}^k V_i(t,x,\omega) \tag{3.5.1}$$

where $b \in \mathcal{VK}$ *and* $p \geq 1$. *Then*

(SMB*) *of* (3.2.1) *implies* (SMB) *of* (3.1.1)

in the context of (H_2').

Proof. Let $\epsilon > 0$ be given and let the trivial solution process of (3.2.1) in the context of (H_2') be stable. Then given $b(\epsilon^p) > 0$ there exist $\delta_\mu^* > 0$ for $\mu = 0, 1$ such that for $\mu = 0, 1$, $\sum_{i=1}^k E[r_{\mu i}] < \delta_\mu^*$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

implies

$$\sum_{i=1}^{k} E[u_i(t,\omega)] < b(\epsilon^p) \qquad \text{for } t \in J, \tag{3.5.2}$$

where $u(t,\omega)$ is any sample solution process of (3.2.1) in the context of (H_2'). We choose

$$\sum_{i=1}^{k} V_i(\mu, x(\mu,\omega), \omega) \le \sum_{i=1}^{k} r_{\mu i} \qquad \text{for } \mu = 0, 1.$$

This together with (3.5.1) and the nature of b, one obtains

$$b(E[\|x(\mu,\omega)\|^p]) \le \sum_{i=1}^{k} E[V_i(\mu, x(\mu,\omega),\omega)] \le \sum_{i=1}^{k} E[r_{\mu i}] \quad \text{for } \mu = 0, 1.$$

This implies that

$$(E[\|x(\mu,\omega)\|^p])^{1/p} \le (b^{-1}(\delta_\mu^*))^{1/p}$$

that is,

$$\|b_\mu\|_p = \|x(\mu,\omega)\|_p \le \delta_\mu \tag{3.5.3}$$

where $\delta_\mu = (b^{-1}(\delta_\mu^*))^{1/p}$ for $\mu = 0, 1$. We claim that with these positive numbers δ_μ in (3.5.3) the stability in the p^{th} moment of the trivial solution process of (3.1.1) holds. If not, there would exist a sample solution process $x(t,\omega)$ and a $t_1 \in J^0 = (0,1)$ such that $\|x(t_1,\omega)\|_p \ge \epsilon$. Setting $m(t,\omega) = V(t, x(t,\omega),\omega)$ and applying the comparison Theorem 3.2.3, we have

$$V(t, x(t,\omega),\omega) \le r(t,\omega) \qquad \text{for } t \in J, \tag{3.5.4}$$

where $x(t,\omega)$ and $r(t,\omega)$ are sample solution and maximal sample solution processes of (3.1.1) and (3.2.1) in the context of (H_2'), respectively. From (3.5.1), (3.5.4) and the nature of b, we get

$$b(\epsilon^p) \le b(E[\|x(t,\omega)\|^p]) \le \sum_{i=1}^{k} E[V_i(t_1, x(t_1,\omega),\omega)] \le \sum_{i=1}^{k} E[r_i(t_1,\omega)].$$

This together with (3.5.2) leads to a contradiction. This contradiction proves that such a t_1 does not exist. Hence the p^{th} moment stability property of the trivial solution process of (3.1.1) in the context of (H_2') is valid.

Remark 3.5.2. We remark that under the hypotheses of Theorem 3.5.1, the trivial solution process of (3.1.1) in the context of (H_2') is stable in probability one provided that the trivial solution process of (3.2.1) in the context of (H_2') is stable in probability one. The verification of this statement follows directly from the proof of Theorem 3.5.1. This is because of the fact that all inequalities are valid with probability one.

By using the integral representation of a solution process of SBVP (3.1.1), sufficient conditions are given for the stability of the trivial solution process. We note that (3.1.1) in the context of coefficients in the boundary conditions as

(H_2'') $\alpha_0(\omega) = \alpha_1(\omega) = 1, \qquad \beta_0(\omega) = \beta_1(\omega) = 0$

$\widehat{\alpha}_0 = \widehat{\alpha}_1 = 1, \qquad \widehat{\beta}_0 = \widehat{\beta}_1 = 0.$

has a unique trivial sample solution process provided $f(t, 0, 0, \omega) \equiv 0$ with$b_\mu = 0$ for $\mu = 0, 1$.

Theorem 3.5.2. *Assume that*

(a) *hypotheses* Theorem 3.1.3 *be satisfied;*

(b) f *in* (3.1.1) *satisfies the following conditions* $f(t, 0, 0, \omega) \equiv 0$ *w.p.* 1*;*

(c) *processes* k_i *for* $i = 1, 2$ *in (b) are independent of* b_μ *for* $\mu = 0, 1$*;*

(d) *for* $p \geq 1$, $1/(1 - \|H(\omega)\|) \in L^p[\Omega, R]$.

Then the trivial solution process of (3.1.1) *is stable in the* p^{th} *moment.*

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Proof. Under the hypotheses of the theorem, and using (3.1.6), we have

$$\begin{aligned}\|y(t,\omega)\| &\leq \|\psi(t,\omega)\| + \int_0^1 |G(t,s,\omega)|\|f(s,y(s,\omega),y'(s,\omega))\|ds \\ &\leq \|\psi(t,\omega)\| + \int_0^1 |G(t,s,\omega)|(k_1(s,\omega)\|y(s,\omega)\| \\ &\quad + k_2(s,\omega)\|y'(s,\omega)\|)ds\end{aligned}$$

and

$$\|y'(t,\omega)\| \leq \|\psi'(t,\omega)\| + \int_0^1 |G_t(t,s,\omega)|(k,(s,\omega)\|y(s,\omega)\| + k_2(s,\omega)\|y'(s,\omega)\|)ds.$$

Now by following proof of Theorem 3.1.3, we arrive at

$$\|y(t,\omega)\| \leq \sum_{i=0}^{\infty} \|H(\omega)\|^i \sup_{s\in[0,1]} (\|(M_0(s,\omega), M_1(s,\omega))\|),$$

where

$$M_0(t,\omega) = \|b_0(\omega)(1-t) + b_1(\omega)t\|$$

and

$$M_1(t,\omega) = \| - b_0(\omega) + b_1(\omega)\|.$$

After some algebraic simplification, the above inequality reduces to

$$\|y(t,\omega)\| \leq \left(\frac{1}{1-\|H(\omega)\|}\right)(\|b_0(\omega)\| + \|b_1(\omega)\|) \tag{3.5.5}$$

for all $t \in [0,1]$. Now by taking the p^{th} exponent and then the expectation on both sides of (3.5.5), we have

$$E[\|y(t,\omega)\|^p] \leq E\left[\left(\frac{1}{1-\|H(\omega)\|}\right)^p \left(\|b_0(\omega)\| + \|b_1(\omega)\|\right)^p\right].$$

This together with assumption (c) yields

$$E[\|y(t,\omega)\|^p] \leq E\left[\left(\frac{1}{1-\|H(\omega)\|}\right)^p\right] E\Big[\big(\|b_0(\omega)\| + \|b_1(\omega)\|\big)^p\Big]. \tag{3.5.6}$$

This implies that

$$\|y(t,\omega)\|_p \leq \left\|\frac{1}{1-\|H(\omega)\|}\right\|_p (\|b_0\|_p + \|b_1\|_p) \tag{3.5.7}$$

for all $t \in [0,1]$. Now for any $\epsilon > 0$, we can choose b_μ for $\mu = 0, 1$ so that

$$\|b_\mu\|_p \leq \epsilon/2\bar{H}, \qquad \text{where } \bar{H} = \left\|\frac{2}{1-\|H(\omega)\|}\right\|_p. \tag{3.5.8}$$

By setting $\delta_\mu = \epsilon/2\bar{H}$ and using (3.5.7) and (3.5.8), we have

$$\|y(t,\omega)\|_p < \epsilon \qquad \text{for } t \in [0,1]$$

whenever

$$\|b_\mu\|_p \leq \delta_\mu \qquad \text{for } \mu = 0, 1.$$

This establishes the p^{th} moment stability of the trivial solution (3.1.1). The proof of the theorem is complete.

Remark 3.5.3. We remark that the almost sure stability and stability in probability analysis (Ladde and Lakshmikantham [67]) of the trivial solution process of (3.1.1) can be deduced from relation (3.5.5). We leave the details to the readers.

3.6. ERROR ESTIMATES

We employ the comparison method to derive the error estimate on p^{th} moment deviation of a solution process of (3.1.1) with respect to the solution process of (3.1.2).

Theorem 3.6.1. *Let the hypotheses of* Theorem 3.2.4 *be satisfied. Further, assume that*

$$b(\|x\|^p) \le \sum_{i=1}^{k} V_i(t, x, \omega) \tag{3.6.1}$$

where $b \in \mathcal{VK}$ and $p \ge 1$. Then

$$b(E[\|y(t,\omega) - z(t)\|^p]) \le \sum_{i=1}^{k} E[r_i(t,\omega)], \qquad \text{for } t \in J, \tag{3.6.2}$$

where $y(t,\omega)$ is a sample solution process of (3.1.1)*; $z(t)$ is a sample solution process of* (3.1.2) *with boundary conditions either in* (3.1.1) *or* (3.1.2), *and $r(t,\omega)$ is the maximal sample solution process of* (3.2.1).

Proof. From the conclusion of Theorem 3.2.4, we have

$$\sum_{i=1}^{k} V_i(t, x(t,\omega) - z(t,\omega), \omega) \le \sum_{i=1}^{k} r_i(t,\omega) \qquad \text{for } t \in J.$$

This together with (3.6.1) and the convexity of b implies

$$b(E[\|x(t,\omega) - z(t,\omega)\|^p) \le \sum_{i=1}^{k} E[r_i(r,\omega)] \qquad \text{on } J.$$

This establishes the relation (3.6.2).

The remainder of the section is devoted to the error estimate analysis in the framework of the Green's function method. For this purpose, we need preliminary material.

A lemma similar to Lemma 3.1.1 can be formulated with respect to (DBVP) (3.1.2). In fact

$$m(t) = \widehat{\psi}(t) + \int_0^1 \widehat{G}(t,s)\widehat{f}(s, m(s), m'(s))ds, \tag{3.6.3}$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

where $\widehat{\psi}(t)$ is defined by $\widehat{b}_0 \frac{\partial \widehat{G}}{\partial s}(t,0) - \widehat{b}_1 \frac{\partial \widehat{G}}{\partial s}(t,1)$, and $\widehat{G}(t,s)$ is a Green's function for DBVP corresponding to (3.1.2)

$$m'' = 0, \qquad \widehat{B}_\mu m(\mu) = \widehat{b}_\mu \qquad \mu = 0,1 \tag{3.6.4}$$

defined by

$$\widehat{G}(t,s) = \begin{cases} \widehat{G}_*(t,s) = (\widehat{\alpha}_1 + \widehat{\beta}_1 - \widehat{\alpha}_1 t)(\widehat{\alpha}_0 s + \widehat{\beta})/\widehat{D}, & 0 \le s \le t \le 1 \\ \widehat{G}^*(t,s) = (\widehat{\alpha}_0 t + \widehat{\beta}_0)(\widehat{\alpha}_1 + \widehat{\beta}_1 - \widehat{\alpha}_1 s)/\widehat{D}, & 0 \le s \le t \le 1, \end{cases}$$

where

$$\widehat{D} = -[\widehat{\alpha}_0(\widehat{\alpha}_1 + \widehat{\beta}_1) + \widehat{\beta}_0 \widehat{\alpha}_1].$$

Similarly

$$x(t) = \psi(t,\omega) + \int_0^1 G(t,s,\omega)\widehat{f}(s,x(s),x'(s))ds, \tag{3.6.5}$$

where $\psi(t,\omega)$ is defined in (3.1.6).

For our future references we list the following hypotheses on f and $\widehat{f}$.

(H_3)

$$\|f(t,x,y,\omega) - f(t,u,v,\omega)\| \le k_1(t,\omega)\|x-u\| + k_2(t,\omega)\|y-v\|$$

$$\|f(t,m,v,\omega) - \widehat{f}(t,m,v)\| \le k_3(t,\omega)\|m\| + k_4(t,\omega)\|v\| + k_5(t,\omega)$$

$$\|\widehat{f}(t,m,v)\| \le k_6(t)\|m\| + k_7(t)\|v\| + k_8(t)$$

for (t,x,y,ω), (t,u,v,ω) and $(t,m,v,\omega) \in J \times R^n \times R^n \times \Omega$, where $k_i \in M[J \times \Omega, R_+]$ for $i = 1,2,\ldots,5$ and $k_j \in C[J,R_+]$ for $j = 6,7,8$ such that $k_i(t,\omega)$ $i = 1,\ldots,5$ are sample Lebesgue integrable processes.

(H_4) Further suppose that the random linear operator $H(\omega)$ defined on $C[J,R] \times C[J;R]$ into itself by

$$H(\omega)(u,v)(t) = \begin{bmatrix} \int_0^1 |G(t,s,\omega)|h(s,u,(s),v(s),\omega)ds \\ \int_0^1 |G_t(t,s,\omega)|h(s,u(s),v(s),\omega)ds \end{bmatrix}^T \tag{3.6.6}$$

has a spectral radius less than one, where $h(s,u,v,\omega) = k_1(t,\omega)u + k_2(t,\omega)v$, k_1 and k_2 are as defined in (H_3).

Theorem 3.6.2. *Assume that the hypothesis* (H_1)–(H_4) *are satisfied. Then*

$$(\|y(t,\omega)-m(t)\|, \|y'(t,\omega)-m'(t)\|) \leq \sum_{i=0}^{\infty}(H(\omega))^i(M_0(\cdot,\omega), M_1(\cdot,\omega)), \tag{3.6.7}$$

where

$$\begin{aligned} M_0(t,\omega) &= |\psi(t,\omega) - \widehat{\psi}(t)| \\ &+ \int_0^1 |G(t,s,\omega)|\Big(k_3(s,\omega)\|m(s)\| + k_4(s,\omega)\|m'(s)\| + k_5(s,\omega)\Big)ds \\ &+ \int_0^1 |G(t,s,\omega) - \widehat{G}(t,s)|\Big(k_6(s)\|m(s)\| + k_7(s)\|m'(s)\| + k_8(s)\Big)ds \end{aligned} \tag{3.6.8}$$

and

$$\begin{aligned} M_1(t,\omega) &= \|\psi'(t,\omega) - \widehat{\psi}'(t)\| \\ &+ \int_0^1 |G_t(t,s,\omega)|\Big(k_3(s,\omega)\|m(s)\| + k_4(s,\omega)\|m'(s)\| + k_5(s,\omega)\Big)ds \\ &+ \int_0^1 |G_t(t,s,\omega) - \widehat{G}_t(t,s)|\Big(k_6(s)\|m(s)\| + k_7(s)\|m'(s)\| + k_8(s)\Big)ds. \end{aligned} \tag{3.6.9}$$

Proof. From (3.1.6) and (3.6.3), we obtain

$$\begin{aligned} y(t,\omega) - m(t) &= \psi(t,\omega) - \widehat{\psi}(t) \\ &+ \int_0^1 G(t,s,\omega)\Big(f(s,y(s,\omega),y'(s,\omega),\omega) - f(s,m(s),m'(s),\omega)\Big)ds \\ &+ \int_0^1 G(t,s,\omega)\Big(f(s,m(s),m'(s),\omega) - \widehat{f}(s,m(s),m'(s))\Big)ds \\ &+ \int_0^1 \Big(G(t,s,\omega) - \widehat{G}(t,s,)\Big)\widehat{f}(s,m(s),m'(s))ds. \end{aligned}$$

Therefore

$$\|y(t,\omega) - m(t)\| \leq \|\psi(t,\omega) - \widehat{\psi}(t)\|$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

$$
\begin{aligned}
&+\int_0^1 |G(t,s,\omega)|\|f(s,y(s,\omega),y'(s,\omega),\omega)-f(s,m(s),m'(s),\omega)\|ds\\
&+\int_0^1 |G(t,s,\omega)|\|f(s,m(s),m'(s),\omega)-\widehat{f}(s,m(s),m'(s))\|ds\\
&+\int_0^1 |G(t,s,\omega)-\widehat{G}(t,s)|\|\widehat{f}(s,m(s),m'(s))\|ds\\
&\le \|\psi(t,\omega)-\widehat{\psi}(t)\|\\
&+\int_0^1 |G(t,s,\omega)|\Big(k_1(s,\omega)\|y-m\|+k_2(s,\omega)\|y'-m'\|\Big)ds\\
&+\int_0^1 |G(t,s,\omega)|\Big(k_3(s,\omega)\|m(s)\|+k_4(s,\omega)\|m'(s)\|+k_5(s,\omega)\Big)ds\\
&+\int_0^1 |G(t,s,\omega)-\widehat{G}(t,s)|\Big(k_6(s)\|m(s)\|+k_7(s)\|m'(s)\|+k_8(s)\Big)ds.
\end{aligned}
\tag{3.6.10}
$$

Similarly

$$
\begin{aligned}
y'(t,\omega)-m'(t)&=\psi'(t,\omega)-\widehat{\psi}'(t)\\
&+\int_0^1 G_t(t,s,\omega)\Big(f(s,y(s,\omega),y'(s,\omega),\omega)-f(s,m(s),m'(s),\omega)\Big)ds\\
&+\int_0^1 G_t(t,s,\omega)\Big(f(s,m(s),m'(s),\omega)-\widehat{f}(s,m(s),m'(s))\Big)ds\\
&+\int_0^1 \Big(G_t(t,s,\omega)-\widehat{G}_t(t,s)\Big)\widehat{f}(s,m(s),m'(s))ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|y'(t,\omega)-m'(t)\|&\le \|\psi'(t,\omega)-\widehat{\psi}'(t)\|\\
&+\int_0^1 |G_t(t,s,\omega)|\|f(s,y(s,\omega),y'(s,\omega),\omega)-f(s,m(s),m'(s),\omega)\|ds\\
&+\int_0^1 |G_t(t,s,\omega)|\|f(s,m(s),m'(s),\omega)-\widehat{f}(s,m(s),m'(s))\|ds\\
&+\int_0^1 |G_t(t,s,\omega)-\widehat{G}_t(t,s)|\|\widehat{f}(s,m(s),m'(s))\|ds\\
&\le \|\psi'(t,\omega)-\widehat{\psi}'(t)\|
\end{aligned}
$$

$$+\int_0^1 |G_t(t,s,\omega)|\Big(k_1(s,\omega)\|y-m\| + k_2(s,\omega)\|y'-m'\|\Big)ds$$
$$+\int_0^1 |G_t(t,s,\omega)|\Big(k_3(s,\omega)\|m(s)\| + k_4(s,\omega)\|m'(s)\| + k_5(s,\omega)\Big)ds$$
$$+\int_0^1 |G_t(t,s,\omega) - \widehat{G}_t(t,\omega)|\Big(k_6(s)\|m(s)\| + k_7(s)\|m'(s)\| + k_8(s)\Big)ds. \tag{3.6.11}$$

From (3.6.10), (3.6.11) and (H_4), we obtain

$$\begin{aligned}(\|y(t,\omega)-m(t)\|, \|y'(t,\omega)-m'(t)\|)&\\ \le H(\omega)(\|y(\cdot,\omega)-m(\cdot)\|, \|y'(\cdot,\omega)-m'(t)\|)(t)&\\ +(M_0(\cdot,\omega), M_1(\cdot,\omega))(t)&\end{aligned} \tag{3.6.12}$$

which implies

$$(I-H(\omega))(\|y(t,\omega)-m(t)\|, \|y'(t,\omega)-m'(t)\|) \le (M_0(\cdot,\omega), M_1(\cdot,\omega))(t).$$

Hence

$$\begin{aligned}(\|y(t,\omega)-m(t)\|, \|y'(t,\omega)-m'(t)\|)&\\ \le (I-H(\omega))^{-1}(M_0(\cdot,\omega), M_1(\cdot,\omega))(t)&\\ \le \sum_{i=1}^{\infty}(H(\omega))^i(M_0(\cdot,\omega), M_1(\cdot,\omega))(t).&\end{aligned}$$

This completes the proof of the theorem.

Remark 3.6.1. For the choice of $\alpha_0(\omega) = \alpha_1(\omega) = 1$ and $\beta_0(\omega) = \beta_1(\omega) = 0$, it is obvious that $\widehat{\alpha}_0 = \widehat{\alpha}_1 = 1$, $\widehat{\beta}_0 = \widehat{\beta}_1 = 0$, that is, hypothesis ($H_2''$),

$$G(t,s,\omega) = \widehat{G}(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1 \\ (1-s)t, & 0 \le t \le s \le 1 \end{cases}$$

$$G_t(t,s,\omega) = \widehat{G}_t(t,s) = \begin{cases} -s, & 0 \le s \le t \le 1 \\ (1-s), & 0 \le t \le s \le 1 \end{cases}$$

$$\psi(t,\omega) = b_0(\omega)(1-t) + b_1(\omega)t, \qquad \widehat{\psi}(t) = \widehat{b}_0(1-t) + \widehat{b}_1 t,$$

$$\psi_t(t,\omega) = -b_0(\omega) + b_1(\omega), \qquad \text{and} \qquad \widehat{\psi}_t(t) = -\widehat{b}_0 + \widehat{b}_1.$$

In this case (3.6.8) and (3.6.9) reduce to

$$M_0(t,\omega) = \|(b_0(\omega) - \widehat{b}_0)(1-t) + (b_1(\omega) - \widehat{b}_1)t\|$$
$$+ \int_0^1 |\widehat{G}(t,s)| \Big(k_3(s,\omega)\|m(s)\| + k_4(s,\omega)\|m'(s)\| + k_5(s,\omega)\Big) ds \quad (3.6.13)$$

and

$$M_1(t,\omega) = \|(-b_0(\omega) + \widehat{b}_0) + (b_1(\omega) - \widehat{b}_1)\|$$
$$+ \int_0^1 |\widehat{G}_t(t,s)| \Big(k_3(s,\omega)\|m(s)\| + k_4(s,\omega)\|m'(s)\| + k_5(s,\omega)\Big) ds \quad (3.6.14)$$

respectively.

Remark 3.6.2. A remark similar to Remark 2.5.4 in the context of Remark 3.5.3 can be formulated. We leave the details to the reader.

3.7. RELATIVE STABILITY

This section, we introduce the concept of relative stability for stochastic boundary value problem (3.1.1) relative (3.1.2) in the context of (H_2').

The concept of relative stability is formulated as follows.

Definition 3.7.1. The two systems of differential equations (3.1.1) and (3.1.2) are said to be (RMB) relatively stable in p^{th} moment, if for each $\epsilon > 0$ and $p \geq 1$, there exist positive real numbers such that for $\mu = 0, 1$, the inequality $\|b_\mu - \widehat{b}_\mu\|_p \leq \delta_\mu$ implies

$$\|y(t,\omega) - m(t)\|_p < \epsilon, \qquad \text{for } t \in J.$$

We present a relative stability result in the context of comparison method.

Theorem 3.7.1. *Let the hypotheses of* Theorem 3.6.1 *be satisfied. Then*

(SMB*) *of* (3.2.1) *implies* (RMB) *of* (3.1.1) *and* (3.1.2).

Proof. Let $\epsilon > 0$ be given. From the stability in the mean of the trivial solution of (3.2.1) in the context of (H_2') and following the argument used in the proof of Theorem 3.5.1, one obtains the inequality (3.5.2). We choose

$$\sum_{i=1}^{k} V_i(\mu, y(\mu,\omega) - m(\mu), \omega) \le \sum_{i=1}^{k} r_{\mu i} \qquad \text{for } \mu = 0, 1.$$

This together with (3.6.1) and the convexity of b, one arrives at

$$\begin{aligned} b(E[\|y(\mu,\omega) - m(\mu)\|^p]) &\le \sum_{i=1}^{k} E[V_i(\mu, y(\mu,\omega) - m(\mu), \omega)] \\ &\le \sum_{i=1}^{k} E[r_{\mu i}], \qquad \mu = 0, 1. \end{aligned}$$

The rest of the proof of the theorem can be formulated by imitating the proof of Theorem 3.5.1. The details are left to the reader. This concludes the proof of the theorem.

Throughout this section hypotheses (H_3) and (H_4) in Section 3.6 are replaced by the following:

(H_3') $\|f(t,x,y,\omega) - f(t,u,v,\omega)\| \le k_1(t,\omega)\|x-u\| + k_2(t,\omega)\|y-v\|$ $\|f(t,m,m',\omega) - \widehat{f}(t,m,m')\| \le k_3(t,\omega)\|m(s)\| + k_4(t,\omega)\|m'(t)\| + k_5(t,\omega)$ for (t,x,y,ω), (t,u,v,ω) and $(t,m,m',\omega) \in J \times R^n \times R^n \times \Omega$ where for $i = 1, 2, 3, 4$, and 5, $k_i \in M[J \times \Omega, R_+]$ such that $k_i(t,\omega)$ are sample Lebesgue integrable processes.

(H_5) For $p \ge 1$, $H(\omega)$ in (3.6.6) satisfies the following condition $E[(1/(1 - \|H(\omega)\|))^{2p}] < \infty$.

Remark 3.7.1. Under the hypothesis of (H_2'') and (H_3') in the context of Remark 3.6.1, relations (3.6.8) and (3.6.9) become

$$M_0(t,\omega) = \|(b_0(\omega) - \widehat{b}_0)(1-t) + (b_1(\omega) - \widehat{b}_1)t\|$$
$$+ \int_0^1 |\widehat{G}(t,s)|(k_3(s,\omega)\|m(s)\| + k_4(s,\omega)\|m'(s)\| + k_5(s,\omega))ds \quad (3.7.1)$$

and

$$M_1(t,\omega) = \|(\widehat{b}_0 - b_0(\omega)) + (b_1(\omega) - \widehat{b}_1)\|$$
$$+ \int_0^1 |\widehat{G}_t(t,s)|(k_3(s,\omega)\|m(s)\| + k_4(s,\omega)\|m'(s)\| + k_5(s,\omega))ds. \quad (3.7.2)$$

Theorem 3.7.2. *Assume that the hypothesis* (H_1), (H_2), (H_2''), (H_3'), (H_4) *and* (H_5) *are satisfied. Further assume that the $2p^{th}$ moment of $M_0(t,\omega)$ and $M_1(t,\omega)$ in* (3.7.1) *and* (3.7.2)*, respectively, can be made arbitrarily small. Then the systems* (3.1.1) *and* (3.1.2) *are relatively stable in the p^{th} moment.*

Proof. From (3.6.7), we have

$$(\|y(t,\omega) - m(t)\|, \|y'(t,\omega) - m'(t)\|) \le \sum_{i=0}^{\infty}(H(\omega))^i(M_0(\cdot,\omega), M_1(\cdot,\omega))(t), \quad (3.7.3)$$

where $M_0(\cdot,\omega)$ and $M_1(\cdot,\omega)$ are defined in (3.7.1) and (3.7.2) respectively. Since $\|H(\omega)\| < 1$, relation (3.7.3) yields

$$\|y(t,\omega) - m(t)\| \le \left(\frac{1}{1 - \|H(\omega)\|}\right)\|M_0(\cdot,\omega), M_1(\cdot,\omega)\|$$
$$\le \left(\frac{1}{1 - \|H(\omega)\|}\right)\Big(M_0(\cdot,\omega) + M_1(\cdot,\omega)\Big). \quad (3.7.4)$$

By taking p^{th} exponent and the mean of both sides of (3.7.4) then using the Hölder inequality (Theorem A.1.6), we get

$$E\left[\|y(t,\omega) - m(t)\|^p\right] \le E\left[\left[\left(\frac{1}{1 - \|H(\omega)\|}\right)(M_0(t,\omega) + M_1(t,\omega))\right]^p\right]$$
$$\le \left(E\left[\left(\frac{1}{1 - \|H(\omega)\|}\right)^{2p}\right]\right)^{1/2}\left(E\left[[\|M_0\|_0 + \|M_1\|_0]^{2p}\right]\right)^{1/2},$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

and hence

$$\|y(t,\omega)-m(t)\|_p \le \left(E\left[\left(\frac{1}{1-\|H(\omega)\|}\right)^{2p}\right]\right)^{1/2p}\left(\left(E[\|M_0\|_0^{2p}]\right)^{1/2p} + \left(E[\|M_1\|_0^{2p}]\right)^{1/2p}\right), \quad (3.7.5)$$

where $\|M_i\|_0 = \max\limits_{t\in[0,1]} M_i(t,\omega)$ for $i = 0, 1$, from the hypothesis (H_5) and the assumption that the $2p^{th}$ moments of $M_0(t,\omega)$ and $M_1(t,\omega)$ are arbitrarily small. Thus conclusion of the theorem follows from (3.7.5), immediately.

Example 3.7.1. We illustrate the above Theorem 3.7.2 for a specific expression of $f(t,x,y,\omega)$. Let

$$f(t,u,v,\omega) = h(t,u,v) + z(t,\omega)H(u)$$

$$\widehat{f}(t,u,v) = h(t,u,v) + \widehat{z}(t)H(u)$$

where $z \in M[J\times\Omega, R]$ and $\widehat{z} = E[z(t,\omega)]$. Then

$$\begin{aligned}
y(t,\omega) - m(t) &= \psi(t,\omega) - \widehat{\psi}(t) \\
&\quad + \int_0^1 G(t,s,\omega)[h(s,y(s,\omega),y'(s,\omega)) + z(s,\omega)H(y(s,\omega))]ds \\
&\quad - \int_0^1 G(t,s,\omega)[h(s,m(s),m'(s)) + z(s,\omega)H(m(s))]ds \\
&\quad + \int_0^1 G(t,s,\omega)[h(s,m(s),m'(s)) + z(s,\omega)H(m(s))]ds \\
&\quad - \int_0^1 \widehat{G}(t,s)[h(s,m(s),m'(s)) + \widehat{z}(s)H(m(s))]ds \\
&= \psi(t,\omega) - \widehat{\psi}(t) + \int_0^1 G(t,s,\omega)[h(s,y(s,\omega),y'(s,\omega)) - h(s,m(s),m'(s)) \\
&\quad + z(s,\omega)(H(y(s,\omega)) - H(m(s)))]ds \\
&\quad + \int_0^1 [G(t,s,\omega) - \widehat{G}(t,s)][h(s,m(s),m'(s))]ds \\
&\quad + \int_0^1 [G(t,s,\omega)z(s,\omega) - \widehat{G}(t,s)\widehat{z}(s)]H(m(s))ds.
\end{aligned}$$

From the assumption of (H_3') and noting the fact that $\widehat{G}(t,s) = G(t,s,\omega)$ as illustrated in Remark 3.6.1, $M_0(t,\omega)$ and $M_1(t,\omega)$ are expressed as follows.

$$M_0(t,\omega) = \|(b_0(\omega) - \widehat{b}_0)(1-t) + (b_1(\omega) - \widehat{b}_1)t\| + \int_0^1 |\widehat{G}(t,s)|(k_4(s,\omega)\|m(s)\|)ds,$$

and

$$M_1(t,\omega) = \|(-b_0(\omega) + \widehat{b}_0) + (b_1(\omega) - \widehat{b}_1)\| + \int_0^1 |\widehat{G}_t(t,s)|(k_4(s,\omega)\|m(s)\|)ds,$$

where $k_4(s,\omega) = c|z(s,\omega) - \widehat{z}(s)|$, where c depends on H.

Remark 3.7.2. We note that the smallness assumption about the M_0 and M_1 in Theorem 3.7.2 seems to be restrictive. However, this assumption is feasible. For example, the stability of DBVP (3.1.2) and the smallness of $E[\|\psi(t,\omega) - \widehat{\psi}(t)\|^p]$ imply the arbitrarily smallness of M_0 and M_1. In this context,the relative stability concept of (3.1.1) relative to (3.1.2) can also be termed as the relative stability of (3.1.1) relative (3.1.2) under constant perturbations.

Remark 3.7.3. A remark similar to Remark 2.6.1 can be formulated to SBVP (3.1.1) with respect to DBVP (3.1.2).

3.8. APPLICATIONS TO PHYSICAL SYSTEMS

a) SLIDER AND RIGID ROLLER BEARING PROBLEMS

In this section, we formulate and analyze the problem of slider bearing and rigid roller bearing in the theory of lubrication. The purpose of lubrication is to separate two surfaces sliding past each other

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

with a film of gas or liquid material which can be sheared without causing any damage to the surfaces. The process of sliding should occasion as little frictional resistance as possible. The lubricant film thickness and dimensions of the bearing surfaces are chosen to insure that no contact occurs between the opposing surfaces. This type of lubrication, which occurs in most journal bearings and thrust bearings, is known as full fluid or hydrodynamic lubrication. The wedge action of a lubricant as postulated by Reynolds will be presented. In particular, the Reynolds' equation in one dimension, which describes the formation of pressure in converging wedge-shaped film, is derived and analyzed.

The assumptions made in deriving the Reynolds' equation are listed as follows:

(a_1) Body forces are neglected. This means that there is no outside field of force such as gravitational force, magnetic force, etc., acting on the lubricant. The assumption is valid for lubrication with nonconducting materials.

(a_2) The pressure is assumed to be constant through a sufficiently small thickness of the film. For example, as the oil film is very thin (of the order of a few thousandths of an inch), the pressure cannot vary significantly across it.

(a_3) The curvatures of the bearing surfaces are considered to be large compared with the film thickness. This assumption reflects the fact that the surface velocities need not be taken as varying in direction.

(a_4) There is no slip at the boundaries. This means that the velocity of the surface is the same as the velocity of the final, adjacent, layer of lubricant.

(a_5) The lubricant is Newtonian, that is, the lubricant obeys the following hypothesis: 'The resistance arising from the want of lubricity in the parts of a fluid, is, other things being equal, proportional to the velocity with which the parts of the fluid are separated from one another.' In this system the parts were separated into layers of equal thickness, so the shear stress, τ, is proportional to the rate of shear (velocity/distance $= \partial u/\partial z$).

(a_6) The flow is laminar. This assumption makes sense for the bearings of normal dimensions and speeds.

(a_7) The fluid inertia is neglected.

(a_8) The viscosity is taken as constant through the thickness of the film. This is rather drastic assumption which simplifies the mathematical analysis.

(a_9) A bearing is assumed to be infinitely long in the y direction. In this case, the lubricant flow in this direction will be zero.

(a_{10}) The coefficient of viscosity of a lubricant is constant.

(a_{11}) It is assumed that the fluid is a compressible gas.

(a_{12}) The bearing is isothermal.

(a_{13}) The bearing roughness is assumed to be in the direction transverse to the sliding or rolling. In this case the asperities on both lubricating surfaces are straight ridges perpendicular to the direction of rolling.

(a_{14}) It is further assumed that there are enough number of asperities within Hertzian zone such that the local average film thickness can be considered as independent of time.

(a_{15}) Assume that the velocities u_1 and u_2 of upper and lower surfaces, respectively, are constants.

Now we are ready to start the derivation of Reynolds' equation

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

governing the pressure in an infinitely long isothermal compressible slider or rigid roller bearing. The derivation of this equation involves the equilibrium of a small element of lubricant, Newton's law of viscous flow, the condition of continuity of flow, and a knowledge of the speeds of the bounding surfaces.

(i) Equilibrium of an element

Consider the problem of the equilibrium of a small element of lubricant with sides of length $\Delta x, \Delta y$ and Δz on which pressures and shear stresses act.

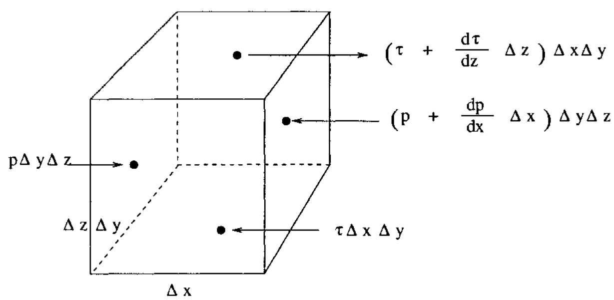


Fig. 3.8.1. Equilibrium of an element.

The forces acting on the element in the x direction are as follows: $p\Delta y\Delta z$ is a force acting on the left-hand face, where p is a pressure; on the right-hand face the pressure is $p+(\partial p/\partial x)\Delta x$ and a corresponding force $(p+(\partial p/\partial x)\Delta x)\Delta y\Delta z$, where the pressure gradient $\partial p/\partial x$ is free to take any value and sign; on the bottom face there is a shear stress τ, and the shear force on this face is $\tau\Delta y\Delta x$; on the top face the shear stress is $\tau+(\partial\tau/\partial z)\Delta z$ and the shear force is $(\tau+(\partial\tau/\partial z)\Delta z)\Delta x\Delta y$. For equilibrium of an element, these forces must balance each other. By equating the forces acting to the right to those acting towards the left, we have

$$p\Delta y\Delta z + \left(\tau + \frac{\partial \tau}{\partial z}\Delta z\right)\Delta x\Delta y = \tau\Delta x\Delta y + \left(p + \frac{\partial p}{\partial x}\Delta x\right)\Delta y\Delta z$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

which implies that

$$\frac{\partial \tau}{\partial z} = \frac{\partial p}{\partial x}. \tag{3.8.1}$$

We note that we should have written the shear stress τ as τ_{xz}, where τ_{xz} denotes the shear stress acting in the x direction and on the plane whose normal is in the z direction. In the same way, it is easy to show that

$$\frac{\partial p}{\partial y} = \frac{\partial \tau_{yz}}{\partial z}. \tag{3.8.2}$$

By assumption (a_2), the pressure gradient in the direction of z is zero, that is,

$$\frac{\partial p}{\partial z} = 0. \tag{3.8.3}$$

From assumption (a_5), we obtain

$$\tau_{xz} = \eta \frac{\partial u}{\partial z} \qquad \text{and} \qquad \tau_{yz} = \eta \frac{\partial v}{\partial z}, \tag{3.8.4}$$

where u and v are the particle velocities in the x and y directions, respectively, η is the coefficient of viscosity. From (3.8.1), (3.8.2) and (3.8.4), we have

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial z}\left(\eta \frac{\partial u}{\partial z}\right) \qquad \text{and} \qquad \frac{\partial p}{\partial y} = \frac{\partial}{\partial z}\left(\eta \frac{\partial v}{\partial z}\right). \tag{3.8.5}$$

From (3.8.3), we can conclude that the pressure is independent of z. Therefore, we can integrate $\frac{\partial p}{\partial x} = \frac{\partial}{\partial z}\left(\eta \frac{\partial u}{\partial z}\right)$ both sides with respect to z. Hence

$$\left(\frac{\partial p}{\partial x}\right) z + c_1 = \eta \frac{\partial u}{\partial z},$$

where c_1 is a constant of integration. This together with assumption (a_{10}) yields

$$u = \left(\frac{1}{\eta}\frac{\partial p}{\partial x}\right)\frac{z^2}{2} + c_1 z + c_2,$$

where c_2 is another constant of integration. To compute c_1 and c_2, we use assumption (a_4). Let H be a local film thickness consisting of the following three parts $H = h + \delta_1 + \delta_2$, so at $z = H$, $u = u_1$, and at $z = 0$, $u = u_2$, where u_1 and u_2 are the velocities of the upper and lower surfaces; h is the local average film thickness; δ_1 and δ_2 are the roughness profiles of the upper and lower surfaces measured from the mean level of surface profiles. Using these conditions and computing c_1 and c_2, we obtain

$$u = \frac{1}{2\eta}\frac{\partial p}{\partial x}(z^2 - zH) + (u_1 - u_2)\frac{z}{H} + u_2. \tag{3.8.6}$$

The rate of lubricant flow in the direction of x per unit width in the direction of y is the integral

$$\begin{aligned} q_x &= \int_0^H u dz = \int_0^H \left[\frac{1}{2\eta}\frac{\partial p}{\partial x}(z^2 - Hz) + (u_1 - u_2)\frac{z}{H} + u_2\right] dz \\ &= -\frac{H^3}{12\eta}\frac{\partial p}{\partial x} + (u_1 + u_2)\frac{H}{2}. \end{aligned} \tag{3.8.7}$$

Similarly, the rate of lubricant in the direction of y per unit width in the direction of x is

$$q_y = -\frac{H^3}{12\eta}\frac{\partial p}{\partial y} + (u_1 + u_2)\frac{H}{2}.$$

This together with assumption (a_9) implies that

$$q_y = 0, \tag{3.8.8}$$

which implies that $\partial p/\partial y = 0$.

(ii) Continuity of flow of column

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

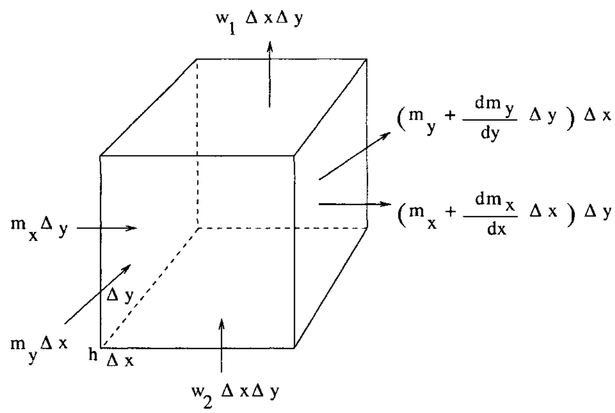


Fig. 3.8.2. Continuity of flow of a column

Consider two surfaces separated by a distance H with a column of base $\Delta x \Delta y$, where H is also the local film thickness. In any fluid the mass flowing into a given volume must be equal to the mass flowing out, whenever the volume does not contain any 'sources' or 'sinks.' Let the rate of mass flow into the left-hand face, in the direction of x, be $m_x = \rho q_x$, where ρ is the density and q_x is the rate of volume flow per unit width in the direction of y. This means that the amount of mass flowing into the column in unit time is $m_x \Delta y$, as Δy is the width of the column. The amount flowing out in the direction of x is $(m_x + (\partial m_x/\partial x)\Delta x)\Delta y$, where $\partial m_x/\partial x$ is the rate of change of flow in the x direction. Similarly, $m_y = \rho q_y$ is the rate of mass flowing in the direction of y per unit width in the x direction, and q_y is the rate of volume flow per unit width in the direction of x. The outflow is $(m_y + (\partial m_y/\partial y)\Delta y)\Delta x$. The rates of inflow in the direction of x and y are

$$m_x \Delta y - \left(m_x + \frac{\partial m_x}{\partial x}\Delta x\right)\Delta y \quad \text{and} \quad m_y \Delta x - \left(m_y + \frac{\partial m_y}{\partial y}\Delta y\right)\Delta x,$$

respectively. Thus the total inflow is the sum of these two, that is,

$$-\left(\frac{\partial m_x}{\partial x}+\frac{\partial m_y}{\partial y}\right)\Delta x\Delta y=-\left(\frac{\partial(\rho q_x)}{\partial x}+\frac{\partial(\rho q_y)}{\partial y}\right)\Delta x\Delta y. \quad (3.8.9)$$

The derivation assumes that the density is constant over the whole column. This follows from assumptions (a_2) and (a_{12}). If the roof and floor of the column are moving at velocities w_1 and w_2 respectively, then the mass content changes at a rate of $\rho(w_1-w_2)\Delta x\Delta y$. We note that w_1-w_2 = rate of change of height H, that is, $w_1-w_2=\partial H/\partial t$. From (3.8.9), we have

$$\frac{\partial(\rho q_x)}{\partial x}+\frac{\partial(\rho q_y)}{\partial y}+\rho\,\frac{\partial H}{\partial t}=0. \quad (3.8.10)$$

This, together with (3.8.8), gives

$$\frac{\partial(\rho q_x)}{\partial x}+\rho\,\frac{\partial H}{\partial t}=0. \quad (3.8.11)$$

In this case, equation (3.8.10) is referred as the continuity equation. From assumption (a_{11}), the gas law states that pv = constant or that p = (constant) ρ. From this, together with (3.8.7), (3.8.8) and assumptions (a_8) and (a_5), equation (3.8.11) becomes

$$\frac{\partial}{\partial x}\left(H^3p\,\frac{\partial p}{\partial x}\right)=6\eta(u_1+u_2)\frac{\partial}{\partial x}(pH)+12\eta p\,\frac{\partial H}{\partial t}. \quad (3.8.12)$$

From (a_{13}) and (a_{15}), the expression of H becomes

$$H=h+\delta_1(x-u_1t)+\delta_2(x-u_2t).$$

This together with

$$\frac{\partial H}{\partial t}=\frac{\partial h}{\partial t}-u_1\,\frac{\partial \delta_1}{\partial x}-u_2\,\frac{\partial \delta_2}{\partial x} \quad (3.8.13)$$

(3.8.12) reduces to

$$\frac{\partial}{\partial x}\left(H^3 p\,\frac{\partial p}{\partial x}\right) = 6\eta(u_1+u_2)\frac{\partial}{\partial x}\,(H^* p) + 12\eta p\,\frac{\partial h}{\partial t}, \qquad (3.8.14)$$

whenever $u_1\,\frac{\partial \delta_1}{\partial x} + u_2\,\frac{\partial \delta_2}{\partial x} = 0$,where

$$H^* = h + v(\delta_2 - \delta_1)$$

and $v = (u_1 - u_2)/(u_1 + u_2)$. From (3.8.14) and ($a_{14}$), we have

$$\frac{\partial}{\partial x}\left(H^3 p\,\frac{\partial p}{\partial x}\right) = 6\eta(u_1+u_2)\frac{\partial}{\partial x}\,(H^* p). \qquad (3.8.15)$$

To study the pressure process p (a natural phenomenon marked by gradual change) determined by (3.8.12), we assume without loss of generality that $x \in [0,1]$, that is, the sliding surfaces are of infinite length in the y direction with unit width in the x direction. Furthermore, we assume that $p(0) = p(1) = 1$. Thus the mathematical problem of studying pressure p is formulated as

$$\left.\begin{aligned} &\frac{d}{dx}\left(H^3 p\,\frac{dp}{dx}\right) = \Lambda\,\frac{d}{dx}\,(pH^*) \\ &p(0) = p(1) = 1 \end{aligned}\right\} \qquad (3.8.16)$$

where $\Lambda = 6\eta(u_1+u_2)$. This is a nonlinear two-point boundary value problem which cannot be solved in closed form.

By setting $2u = p^2$, the SBVP (3.8.16) is transformed into the quasilinear problem given by

$$\frac{d}{dx}\left(H^3(x,\omega)\,\frac{du}{dx}\right) = \Lambda\,\frac{d}{dx}\,(H^*(x,\omega)\sqrt{2u}) \qquad (3.8.17)$$

with

$$u(0,\omega) = u(1,\omega) = \frac{1}{2}. \qquad (3.8.18)$$

We note that SBVP (3.8.17) and (3.8.18) is equivalent to

$$u''(x,\omega) = g(x, u(x,\omega), u'(x,\omega), \omega) \tag{3.8.19}$$

with

$$u(0,\omega) = \frac{1}{2} = u(1,\omega), \tag{3.8.20}$$

where

$$g(x,u,u',\omega) = H_1^{-3}(x,\omega)[\Lambda H_2(x,\omega)[2u]^{-1/2} - 3H_1'(x,\omega)H_1^2(x,\omega)]u' + \Lambda H_1^{-3}(x,\omega)H_2'(s,\omega)\sqrt{2u},$$

$H = H_1$ and $H^* = H_2$.

$$\alpha(x,\omega) = \frac{1}{2} \tag{3.8.21}$$

and

$$\beta(x,\omega) = \frac{1}{2}\left[1 + \Lambda \int_0^x H_2(s,\omega)H_1^{-3}(s,\omega)ds\right]^2 \tag{3.8.22}$$

can be selected as lower and upper solution processes of (3.8.19)–(3.8.20). From the nature of the stochastic processes H_2 and H_1, for the above choice of α and β, we have

$$\alpha(x,\omega) \le \beta(x,\omega) \qquad \text{w.p. } 1, \quad x \in I.$$

From the definition of $g(x,u,u',\omega)$, it is obvious that $g(x,u,u',\omega)$ satisfies a Nagumo condition with respect to α and β with a Nagumo function being linear (say, $(L_1(\omega)|u'| + L_2(\omega))$). As a consequence of the Nagumo condition, one can find a positive random number $N(\omega)$ such that

$$|u'(x,\omega)| \le N(\omega) \qquad \text{w.p. } 1, \quad x \in I.$$

We further note that the partial derivative of $g(x,u,u',\omega)$ with respect to u is equal to

$$g_u(x,u,u',\omega) = \Lambda H_1^{-3}(x,\omega)[H_2'(x,\omega)(2u)^{-1/2} - H_2(x,\omega)(2u)^{-3/2}u']$$

and moreover

$$|g_u(x,u,u',\omega)| \le Q(C,\omega) \qquad \text{w.p. 1}$$

whenever

$$u \in [\alpha(x,\omega), \beta(x,\omega)] \qquad \text{w.p. 1}$$

and $|u'| \le C$ where $Q(C,\omega) = \max_{x\in I}\{\Lambda H_1^{-3}(x,\omega)[|H_2'(x,\omega)| + |H_2(x,\omega)|C]\}$. From the regularity condition on H_1 and H_2, we conclude that

$$\frac{Q(C,\omega)}{C^2} \to 0 \qquad \text{w.p. 1} \quad \text{as } C \to \infty$$

and

$$\frac{\text{linear Nagumo function in } s}{s^2} \to \text{finite} \qquad \text{w.p. 1} \quad \text{as } s \to \infty.$$

This together with the application of Lemma 3.2.1, Remarks 3.2.2 and 3.2.4 and Theorem 3.1.2, it follows that SBVP (3.8.19) and (3.8.20) have a unique sample solution process.

For the deterministic boundary value problem

$$(h^3 m')' = \Lambda(h\sqrt{2m})' \tag{3.8.23}$$

and

$$m(0) = m(1) = \frac{1}{2} \tag{3.8.24}$$

corresponding to the smooth surfaces, one can verify the fact that

$$\bar{\alpha}(x) = \frac{1}{2} \qquad \text{and} \qquad \bar{\beta}(x) = \frac{1}{2}\left[1 + \Lambda \int_0^x h^{-2}(s)ds\right]^2 \tag{3.8.25}$$

are lower and upper solutions of the above deterministic boundary value problem. For more details, see Chandra and Davis [16]. We

further note that $\bar{N}$ is the Nagumo number of BVP (3.8.23)–(3.8.24), that is, $|m'(x)| \leq \bar{N}$ on $I = [0,1]$ for any solution of (3.8.23)–(3.8.24).

In the following, we obtain an upper estimate on the deviation

$$\Delta u(x,\omega) = u(x,\omega) - m(x) \tag{3.8.26}$$

where $u(x,\omega)$ is a sample solution process of (3.8.17)–(3.8.18) and $m(x)$ is the solution of the corresponding deterministic boundary value problem (3.8.23)–(3.8.24) in which the surfaces are assumed smooth.

Setting

$$V(x, u-m) = (u-m)^n \tag{3.8.27}$$

for any integer $n \geq 1$, and differentiating $V(x, u-m)$ in (3.8.27) with respect to x, we obtain

$$V'(x, u-m) = n(u-m)^{n-1}(u'-m'); \tag{3.8.28}$$

multiplying the equation (3.8.28) by $H_1^3(x,\omega)$, we get

$$H_1^3(x,\omega)V' = H_1^3(x,\omega)(u'-m')n(u-m)^{n-1}. \tag{3.8.29}$$

Again, by differentiating both sides of the above equation with respect to x, we have

$$\begin{aligned}(H_1^3(x,\omega)V')' = {} & (H_1^3(x,\omega)(u'-m'))'n(u-m)^{n-1} \\ & + n(n-1)H_1^3(x,\omega)(u'-m')^2(u-m)^{n-2}.\end{aligned} \tag{3.8.30}$$

We note that in the above discussion, it is assumed that $n \geq 2$. Otherwise, for $n = 1$, the second term in the right-hand side of (3.8.30) is assumed to be zero. Our objective is to compute the expression in the right-hand side with respect to problems (3.8.17)–(3.8.18) and

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

(3.8.23)–(3.8.24), that is, we will replace $(H_1^3(x,\omega)u')'$ and $(h^3(x)m')'$ by $\Lambda(H_2(x,\omega)\sqrt{2u})'$and $\lambda(h(x)\sqrt{2m})'$, respectively. For this purpose, by adding $\Lambda(H_2(x,\omega)\sqrt{2m})' - \Lambda(H_2(x,\omega)\sqrt{2m})' = 0$ and $(h^3(x)m')' - \Lambda(h(x)\sqrt{2m})' = 0$ to the right-hand side of (3.8.30) and rearranging the terms we get

$$
\begin{aligned}
(H_1^3(x,\omega)V')' &= [(H_1^3(x,\omega)u')' - \Lambda(h(x)\sqrt{2m})' + (h^3(x)m')' \\
&\quad -(H_1^3(x,\omega)m')' + \Lambda(H_2(x,\omega)\sqrt{2m})' - \Lambda(H_2(x,\omega)\sqrt{2m})']n(u-m)^{n-1} \\
&\quad +n(n-1)(u-m)^{n-2}(u'-m')^2 H_1^3(x,\omega) \\
&= [\Lambda(H_2(x,\omega)\sqrt{2u})' - \Lambda(H_2(x,\omega)\sqrt{2m})' \\
&\quad +\Lambda([H_2(x,\omega) - h(x)]\sqrt{2m})' - ([H_1^3(x,\omega) - h^3(x)]m')']n(u-m)^{n-1} \\
&\quad +n(n-1)(u-m)^{n-2}(u'-m')^2 H_1^3(x,\omega) \\
&= [\Lambda(H_2(x,\omega)[\sqrt{2u} - \sqrt{2m}])' + \Lambda([H_2(x,\omega) - h(x)]\sqrt{2m})' \\
&\quad -([H_1^3(x,\omega) - h^3(x)]m')']n(u-m)^{n-1} \\
&\quad +n(n-1)(u-m)^{n-2}(u'-m')^2 H_1^3(x,\omega). \qquad (3.8.31)
\end{aligned}
$$

Since (Lemma A.2.1)

$$
\sqrt{2u} - \sqrt{2m} = \left(\int_0^1 [2(su + (1-s)m)]^{-1/2} ds\right)(u-m), \qquad (3.8.32)
$$

from (3.8.31) and (3.8.32), we get

$$
\begin{aligned}
(H_1^3(x,\omega)V')' &= \\
&\left[\Lambda\left(H_2(x,\omega)\left(\int_0^1 [2(su+(1-s)m)]^{-1/2} ds\right)(u-m)\right)'\right] \\
&\quad \times n(u-m)^{n-1} + F_0(x,\omega), \qquad (3.8.33)
\end{aligned}
$$

where

$$
F_0(x,\omega) = [\Lambda([H_2(x,\omega) - h(x)]\sqrt{2m})'
$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

$$-([H_1^3(x,\omega)-h^3(x)]m')']n(u-m)^{n-1}$$
$$+n(n-1)(u-m)^{n-2}(u'-m')^2H_1^3(x,\omega)$$
$$= nf'(x,\omega)(u-m)^{n-1}+n(n-1)(u-m)^{n-2}(u'-m')^2H_1^3(x,\omega),$$

and

$$f(x,\omega)=\Lambda(H_2(x,\omega)-h(x))\sqrt{2m}-(H_1^3(x,\omega)-h^3(x))m'. \quad (3.8.34)$$

We observe that $F_0(x,\omega)$ and $f'(x,\omega)$ depend on the parameters u,u',m and m'. By performing the differentiation operation in the right-hand side of (3.8.33) and using (3.8.28), we have

$$(H_1^3(x,\omega)V')'=q_1(x,\omega)V'+n(q_1(x,\omega))'V+F_0(x,\omega), \quad (3.8.35)$$

where

$$q_1(x,\omega)=\Lambda H_2(x,\omega)\int_0^1[2(su+(1-s)m)]^{-1/2}ds. \quad (3.8.36)$$

We note that $q_1(x,\omega)$ depends on u and m. Let $\gamma(x,\omega)$ be a stochastic process defined by

$$\sup_{(u,u',m,m')\in\bar{B}}\{n|(q_1(x,\omega))'|\}<\gamma(x,\omega), \quad (3.8.37)$$

where

$$\bar{B}=\{(u,u',m,m')\in R^4:(u,u',m,m')\in[\alpha(x,\omega),\beta(x,\omega)]$$
$$\times[-N(\omega),N(\omega)]\times[\bar{\alpha}(x),\bar{\beta}(x)]\times[-\bar{N},\bar{N}],x\in I\};$$

$\alpha(x,\omega)$, $\beta(x,\omega)$, $\bar{\alpha}(x)$, $\bar{\beta}(x)$, $N(\omega)$ and $\bar{N}$ are as defined before. Now by adding and subtracting $\gamma(x,\omega)V$ to the right-side of (3.8.35), we get

$$(H_1^3(x,\omega)V')'=q_1(x,\omega)V'+(\gamma(x,\omega)+n(q_1(x,\omega))')V$$
$$+F_0(x,\omega)-\gamma(x,\omega)V. \quad (3.8.38)$$

Now, multiplying both sides of (3.8.38) by

$$\exp\left[-\int_0^x q_1(\theta,\omega)H_1^{-3}(\theta,\omega)d\theta\right],$$

rewriting and rearranging some of the terms, we obtain

$$(H(x,\omega)V')' \geq q(x,\omega)V + \lambda(x,\omega), \tag{3.8.39}$$

where

$$H(x,\omega) = H_1^3(x,\omega)\exp\left[-\int_0^x K(\theta,\omega)d\theta\right], \tag{3.8.40}$$

$$q(x,\omega) = (\gamma(x,\omega) + n(q_1(x,\omega))')\exp\left[-\int_0^x K(\theta,\omega)d\theta\right], \tag{3.8.41}$$

$$\lambda(x,\omega) = (F_0(x,\omega) - \gamma(x,\omega)(u-m)^n)\exp\left[-\int_0^x K(\theta,\omega)d\theta\right], \tag{3.8.42}$$

$$K(x,\omega) = q_1(x,\omega)H_1^{-3}(x,\omega), \tag{3.8.43}$$

and $q_1(x,\omega)$ is as defined in (3.8.36). From (3.8.33), (3.8.36), (3.8.41), (3.8.42) and (3.8.43), it is obvious that the coefficients $H(x,\omega)$, $q(x,\omega)$ and $\lambda(x,\omega)$ depend on the parameters (u,u',m,m'). Moreover, they are sample continuous on $\bar{B}$ in (3.8.37).

As noted before, $n=1$ in (3.8.27), the second term in (3.8.30) is zero. Therefore, in this case $F_0(x,\omega)$ in (3.8.33) is equal to $f'(x,\omega)$ in (3.8.34) which depends only on the two parameters m and m'. In the light of this remark, equation (3.8.33) reduces to

$$(H_1^3(x,\omega)V'(x,\omega))' = (q_1(x,\omega)V(x,\omega))' + f'(x,\omega), \tag{3.8.44}$$

where

$$V(x,\omega) = u(x,\omega) - m(x).$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

From (3.8.18), (3.8.24) and (3.8.27), we have

$$V(0, u(0,\omega) - m(0)) = V(1, u(1,\omega) - m(1)) = 0. \tag{3.8.45}$$

By an application of the scalar version of the comparison theorem Theorem 3.2.4 we get

$$[u(x,\omega) - m(x)]^n \le r(x,\omega), \tag{3.8.46}$$

where $u(x,\omega)$ and $m(x)$ are as defined in (3.8.26); $r(x,\omega)$ is the sample solution process of the comparison SBVP

$$(H(x,\omega)r')' = q(x,\omega)r + \lambda(x,\omega) \tag{3.8.47}$$

with

$$r(0,\omega) = r(1,\omega) = 0. \tag{3.8.48}$$

We note that $q(x,\omega)$ and $H(x,\omega) > 0$ w.p. 1, and hence

$$r(x,\omega) = \int_0^1 G(x,s,\omega)\lambda(s,\omega)ds, \tag{3.8.49}$$

where $G(x,s,\omega)$ is the stochastic Green's function associated with SBVP(3.8.47)–(3.8.48). From (3.8.46) and (3.8.49), we have

$$[u(x,1) - m(x)]^n \le \int_0^1 G(x,s,\omega)\lambda(s,\omega)ds. \tag{3.8.50}$$

We further note that $G(x,s,\omega)$ depends continuously (in the sense of sample) on u and m where $\alpha(x,\omega) \le u \le \beta(x,\omega)$ and $\bar{\alpha}(x) \le m \le \bar{\beta}(x)$, respectively. Therefore, from (3.8.50), we obtain

$$[u(x,\omega) - m(x)]^n \le \int_0^1 G^*(x,s,\omega)\lambda^*(s,\omega)ds \qquad \text{w.p. } 1, \tag{3.8.51}$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

where

$$G^*(x,s,\omega) = \sup_{\substack{\alpha \le u \le \beta \\ \bar{\alpha} \le m \le \bar{\beta}}} \{G(x,s,\omega)\} \qquad \text{and} \qquad \lambda^*(s,\omega) = \sup_{\bar{B}}\{\lambda(s,\omega)\}.$$

We note that $G^*(x,s,\omega)$ and $\lambda^*(x,s,\omega)$ are independent of the parameters u and m.

We remark that for $n = 2$, (3.8.51) gives an estimate on the quadratic mean deviation. Other sample estimates on the various types of deviations can be obtained, similarly.

Sample estimates on various kinds of deviations can be obtained, directly, from problems (3.8.17)–(3.8.18) and (3.8.23)–(3.8.24). In order to obtain such estimates, we take $V(x,\omega)$ as defined in (3.8.44), that is, $V(x,\omega) = u(x,\omega) - m(x)$. From this we observe that

$$V(0,\omega) = V(1,\omega) = 0. \tag{3.8.52}$$

By sample integration (3.8.44), we arrive at

$$H_1^3(x,\omega)V' = q_1(x,\omega)V + f(x,\omega) + c_0, \tag{3.8.53}$$

where c_0 is a constant of integration. By employing a method of variation of parameters, a general solution process of (3.8.53) is given by

$$\begin{aligned} V(x,\omega) = \exp\left[\int_0^x K(\theta,\omega)d\theta\right] c_1 \\ + \int_0^x H_1^{-3}(s,\omega)[f(s,\omega)+c_0] \exp\left[\int_s^x K(\theta,\omega)d\theta\right] ds, \end{aligned} \tag{3.8.54}$$

where $K(x,\omega)$ is as defined in (3.8.43). This together with the boundary conditions in (3.8.52) yields

$$V(0,\omega) = c_1 = 0$$

and

$$V(1,\omega) = 0 = c_0 \int_0^1 H_1^{-3}(s,\omega) \exp\left[\int_s^1 K(\theta,\omega)d\theta\right] ds$$
$$+ \int_0^1 f(s,\omega) H_1^{-3}(s,\omega) \exp\left[\int_s^1 K(\theta,\omega)d\theta\right] ds.$$

Hence

$$c_0 = \frac{\int_0^1 f(s,\omega)H_1^{-3}(s,\omega)\exp[\int_s^x K(\theta,\omega)d\theta]ds}{\int_0^1 H_1^{-3}(s,\omega)\exp[\int_s^x K(\theta,\omega)d\theta]ds}.$$

From this, (3.8.53) reduces to

$$V(x,\omega) = \int_0^x f(x,\omega)H_1^{-3}(s,\omega)\exp\left[\int_s^x K(\theta,\omega)d\theta\right] ds$$
$$- \frac{[\int_0^x H_1^{-3}(s,\omega)\exp[\int_s^x K(\theta,\omega)d\theta]ds] \quad [\int_0^1 f(s,\omega)H_1^{-3}(s,\omega)\exp[\int_s^x K(0,\omega)d\theta]ds]}{\int_0^1 H_1^{-3}(s,\omega)\exp[\int_s^x K(\theta,\omega)d\theta]ds}.$$

Splitting each of the integrals from 0 to 1 into two parts, from 0 to x and x to 1, the above equation reduces to

$$V(x,\omega) = \Bigg(\left[\int_0^x f(s,\omega)H_1^{-3}(s,\omega)\exp\left[\int_s^x K(\theta,\omega)d\theta\right]ds\right]$$
$$\cdot \left[\int_x^1 H_1^{-3}(s,\omega)\exp\left[\int_s^x K(\theta,\omega)d\theta\right]ds\right]$$
$$- \left[\int_0^x H_1^{-3}(s,\omega)\exp\left[\int_s^x K(\theta,\omega)d\theta\right]ds\right]$$
$$\cdot \left[\int_x^1 f(s,\omega)H_1^{-3}(s,\omega)\exp\left[\int_s^x K(\theta,\omega)d\theta\right]ds\right]\Bigg) \Big/$$
$$\left(\int_0^1 H_1^{-3}(s,\omega)\exp\left[\int_s^x K(\theta,\omega)d\theta\right]ds\right)$$

which implies that

$$|V(s,\omega)| \le \Bigg(\left[\int_0^x |f(s,\omega)|H_1^{-3}(s,\omega)\exp\left[\int_s^x K(\theta,\omega)d\theta\right]ds\right]$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

$$
\begin{aligned}
&\cdot \left[\int_x^1 H_1^{-3}(s,\omega)\exp\left[\int_s^x K(\theta,\omega)d\theta\right]ds\right]\\
&+\left[\int_0^x H_1^{-3}(s,\omega)\exp\left[\int_s^x K(\theta,\omega)d\theta\right]ds\right]\\
&\cdot \left[\int_x^1 |f(s,\omega)|H_1^{-3}(s,\omega)\exp\left[\int_0^x K(\theta,\omega)d\theta\right]ds\right]\Bigg)\Bigg/\\
&\left(\left[\int_0^1 H_1^{-3}(s,\omega)\exp\left[\int_s^x K(\theta,\omega)d\theta\right]ds\right]\right)\\
\leq &\int_0^1 |f(s,\omega)|H_1^{-3}(s,\omega)\exp\left[\int_s^x K(\theta,\omega)d\theta\right]ds.
\end{aligned}
$$

This together with the definition of $V(s,\omega)$ establishes the following inequality

$$
|u(x,\omega)-m(s)| \leq \int_0^1 |f(s,\omega)|H_1^{-3}(s,\omega)\exp\left[\int_s^x K(\theta,\omega)d\theta\right]ds. \tag{3.8.55}
$$

Moreover, from the definition of $f(x,\omega)$ (in (3.8.34)) and (3.8.55), we have

$$
\begin{aligned}
|u(x,\omega)-m(x)| \leq \Lambda\int_0^1 \Bigg[\Big(&|H_2(s,\omega)-h(s)|\sqrt{2\bar{\beta}(s)}\\
&+|H_1^3(s,\omega)-h^3(s)|h^{-2}(s)v(h(s),\bar{\beta}(s),\bar{N})\Big)H_1^{-3}(s,\omega)\\
&\cdot\exp\left[\int_s^x K(\theta,\omega)d\theta\right]\Bigg]ds \quad \text{w.p. } 1,
\end{aligned} \tag{3.8.56}
$$

where

$$
v(h(s),\bar{\beta}(s),\bar{N}) = \left|\frac{h(x)\sqrt{2\bar{\beta}(s)}-h(0)+\Lambda^{-1}h^3(0)\bar{N}}{h(s)}\right|.
$$

To establish the validity of (3.8.56), we first note from the definition of $f(x,\omega)$ in (3.8.34) that

$$
|f(x,\omega)| \leq \Lambda|H_2(x,\omega)-h(x)|\sqrt{2m(x)}+|H_1^3(x,\omega)-h^3(x)||m'(x)||, \tag{3.8.57}
$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

where $m(x)$ is the solution of (3.8.23)–(3.8.24). From definition of $\bar{B}$, we further note that

$$\bar{\alpha}(x) \leq m(x) \leq \bar{\beta}(x) \qquad \text{for } x \in I, \tag{3.8.58}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are as defined in (3.8.25). Furthermore, by integrating (3.8.23) from 0 to x and using (3.8.24), we obtain

$$h^3(x)m'(x) - h^3(0)m'(0) = \Lambda h(x)\sqrt{2m(x)} - \Lambda h(0)$$

from which we have

$$\begin{aligned} |m'(x)| &\leq \Lambda h^{-2}(x)\left[\frac{h(x)\sqrt{2\bar{\beta}(x)} - h(0) + \Lambda^{-1}h^3(0)\bar{N}}{h(x)}\right] \\ &\leq \Lambda h^{-2}(x)v(h(x), \bar{\beta}(x), \bar{N}), \end{aligned} \tag{3.8.59}$$

in view of the fact that $|m'(x)| \leq \bar{N}$ for all $x \in I$ and (3.8.58), where $v(h(x), \bar{\beta}(x), \bar{N})$ is defined in (3.8.56) and $\bar{N}$ is defined before. From (3.8.57) and (3.8.59), inequality (3.8.55) reduces to (3.8.56).

The sample estimates in (3.8.51) and (3.8.56) will be used to obtain estimates on absolute mean and mean-square deviation. For this purpose, one can take the mean of both sides of the random inequality (3.8.51), we obtain

$$E\left[[u(x,\omega) - m(x)]^n\right] \leq \int_0^1 E\left[G^*(x,s,\omega)\lambda^*(s,\omega)\right]ds.$$

This, together with the application of Schwarz's inequality, yields

$$\begin{aligned} &E\left[[u(x,\omega) - m(x)]^n\right] \leq \\ &\qquad \left[\int_0^1 E\left[(G^*(x,s,\omega))^2\right]ds \int_0^1 E\left[(\lambda^*(s,\omega))^2\right]ds\right]^{1/2}. \end{aligned} \tag{3.8.60}$$

For $n = 2$, (3.8.60) gives estimates on the absolute mean-square deviation of $u(x,\omega)$ from the smooth case.

From the above analysis, we remark that the estimate on the mean of the solution process of the comparison SBVP (3.8.47)–(3.8.48) is the estimate on the absolute mean-square deviation. From (3.8.60), we observe that the estimates on mean-square deviation are not, directly, related to the roughness profile processes; however, they are related through the solution process of (3.8.47)–(3.8.48) whose coefficients are the stochastic processes modeling the roughness profile.

On the other hand, if we employ the sample estimate on the deviation in (3.8.56), we can obtain estimates on the absolute mean and mean-square deviation in terms of the roughness correlation of $\delta_1(x,\omega)+\delta_2(x,\omega)$ and $\delta_1(x,\omega)-\delta_2(x,\omega)$.

To obtain the absolute deviation, we take the mean and then apply Schwarz's inequality to the expression in (3.8.56). Thus we have

$$
\begin{aligned}
E\left[|u(x,\omega)-m(x)|\right] \le \Lambda & \Bigg[\int_0^1 E\Big[\Big(|H_2(s,\omega)-h(s)|\sqrt{2\bar{\beta}(s)} \\
& +|H_1^3(s,\omega)-h^3(s)|h^{-2}(s)v(h(s),\bar{\beta}(s),\bar{N})\Big)^2\Big]ds\Bigg]^{1/2} \\
& \cdot\left[\int_0^1 E\left[H_1^{-6}(s,\omega)\exp\left[2\int_s^x K(\theta,\omega)d\theta\right]\right]\right]^{1/2} \\
\le \sqrt{2}\Lambda & \left[\int_0^1 E\left[H_1^{-6}(s,\omega)\exp\left[\int_s^1 K(\theta,\omega)d\theta\right]\right]ds\right]^{1/2} \\
& \cdot\Bigg[2\int_0^1\Big(E\left[H_2(s,\omega)-h(s)\right]^2 2\bar{\beta}(s) \\
+E\Big[\left(H_1^3(s,\omega)-h^3(s)\right)^2 & h^{-4}(s)v^2(h(s),\bar{\beta}(s),\bar{N})\Big]\Big)ds\Bigg]^{1/2}. \qquad (3.8.61)
\end{aligned}
$$

Since

$$H_1^3(x,\omega)-h^3(x)\le 7|\delta_1(x,\omega)+\delta_2(x,\omega)|h^2(x) \qquad (3.8.62)$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

whenever $|\delta_1(x,\omega)| + |\delta_2(x,\omega)| < h(x)$, from (3.8.61), we have

$$\langle (H_1^3(x,\omega) - h^3(x))^2 \rangle \le 49h^4(x)\langle (H_1(x,\omega) - h(x))^2 \rangle. \qquad (3.8.63)$$

From H_1 and H_2, (3.8.62), (3.8.63), inequality (3.8.61) reduces to

$$E\big[|u(x,\omega) - m(x)|\big] \le \sqrt{2}A\left[2v^2\int_0^1 R_\eta(s)\bar{\beta}(s)ds + 49\int_0^1 R_\xi(s)v^2(h(s),\bar{\beta}(s),\bar{N})ds\right]^{1/2}, \qquad (3.8.64)$$

where

$$A = \Lambda\left[\int_0^1 \left\langle H_1^{-6}(s,\omega)\exp\left[\int_0^1 K(\theta,\omega)d\theta\right]\right\rangle ds\right]^{1/2};$$

$R_\eta(s) = \langle (\delta_1(s,\omega) - \delta_2(s,\omega))^2 \rangle$ and $R_\xi(s) = \langle (\delta_1(s,\omega) + \delta_2(s,\omega))^2 \rangle$. Similarly, squaring the expression (3.8.56), a bound for the absolute mean-square deviation can be obtained, analogously.

In the following, an estimate for the unit mean deviation of a normalized load carrying capacity is obtained.

Let $|\Delta W|$ denote the mean deviation of the load carrying capacity defined as

$$|\Delta W| = \int_0^1 \langle |u(s,\omega) - m(s)| \rangle ds. \qquad (3.8.65)$$

Then, noting (3.8.64), the unit mean deviation is given by

$$\rho = \left|\frac{\Delta W}{W_m}\right| \le \frac{\max_{x\in I=[0,1]}\langle |u(x,\omega) - m(x)| \rangle}{\int_0^1 m(s)ds}. \qquad (3.8.66)$$

Furthermore, by noting (3.8.60), the unit mean-square deviation can be derived, similarly.

We recall that our discussion is with respect to the transformed SBVP (3.8.17)–(3.8.18). However, our main objective is to study

SBVP (3.8.16). Therefore, we need to obtain the estimates on the deviation of $p(x,\omega)$ from the smooth case, that is,

$$(h^3(x)p_0p_0')' = \Lambda(h(x)p_0)' \tag{3.8.67}$$

with

$$p_0(0) = p_0(1) = 1. \tag{3.8.68}$$

Let $p(x,\omega)$, $u(x,\omega)$, $m(x)$ and $p_0(x)$ be solution processes of (3.8.16), (3.8.17)–(3.8.18), (3.8.23)–(3.8.24) and (3.8.67)–(3.8.68), respectively. From the transformation $p^2 = 2u$, we have

$$|p(x,\omega) - p_0(x)| = |\sqrt{2u(x,\omega)} - \sqrt{2m(x)}| \qquad \text{for } x \in I. \tag{3.8.69}$$

On the other hand, from the generalized mean-value theorem Lemma A.2.1, we get

$$\sqrt{2u(x,\omega)} - \sqrt{2m(x)} = \int_0^1 [2(su(x,\omega) + (1-s)m(s))]^{-1/2} ds (u(x,\omega) - m(x)). \tag{3.8.70}$$

Further, we note that

$$\left| \int_0^1 [2(su(x,\omega) + (1-s)m(x))]^{-1/2} ds \right| \leq 1 \qquad \text{w.p. } 1 \tag{3.8.71}$$

whenever $u(x,\omega) \in [\alpha(x,\omega), \beta(x,\omega)]$, $m(x) \in [\bar{\alpha}(x), \bar{\beta}(x)]$ for $x \in I$; α, β, $\bar{\alpha}$ and $\bar{\beta}$ as defined before. From (3.8.69), (3.8.70) and (3.8.71), we have

$$|p(x,\omega) - p_0(x)| \leq |u(x,\omega) - m(x)| \qquad \text{w.p. } 1 \quad \text{for } x \in I. \tag{3.8.72}$$

From (3.8.72), we get

$$E\left[|p(x,\omega) - p_0(x)|\right] \leq E\left[|u(x,\omega) - m(x)|\right]. \tag{3.8.73}$$

From (3.8.72) and (3.8.51) the sample bounds for quadratic mean deviation of $p(x,\omega)$ from $p_0(x)$ is given by

$$|p(x,\omega) - p_0(x)|^2 \leq \int_0^1 G^*(x,s,\omega)\lambda^*(s,\omega)ds \qquad \text{w.p. 1.} \qquad (3.8.74)$$

Similarly, from (3.8.56) the sample estimate for deviation is

$$\begin{aligned} |p(x,\omega) - p_0(x)| \leq \Lambda \int_0^1 \Bigg[\Bigg(& |H_2(s,\omega) - h(s)|\sqrt{2\bar{\beta}(s)} \\ & + H_1^3(s,\omega) - h^3(s)|h^{-2}(s)v(h(s),\bar{\beta}(s),\bar{N}) \Bigg) H_1^{-3}(s,\omega) \\ & \cdot \exp\left[\int_s^x q_1(0,\omega)H_1^{-3}(\theta,\omega)d\theta\right] ds \Bigg]. \qquad (3.8.75) \end{aligned}$$

From (3.8.74) and (3.8.75), the estimates for the absolute mean and mean-square deviation of $p(x,\omega)$ with $p_0(x)$ are given by (3.8.64) and (3.8.60), respectively. From (3.8.64), (3.8.65), (3.8.66), (3.8.72), the estimate for the unit mean-deviation is given by

$$\rho = \frac{\int_0^1 \langle |p(s,\omega) - p_0(s)| \rangle ds}{\int_0^1 p_0(s)ds} \leq \frac{\max_{x\in I}\langle |p(x,\omega) - p_0(x)| \rangle}{\int_0^1 p_0(s)ds},$$

$$\rho \leq \frac{\sqrt{2}A\Lambda\left[2v^2\int_0^1 R_\eta(s)\bar{\beta}(s)ds + 49\int_0^1 R_\xi(s)v^2(h(s),\bar{\beta}(s),\bar{N})ds\right]}{\int_0^1 p_0(s)ds}. \qquad (3.8.76)$$

The above inequality estimates explicitly the effects of the roughness profiles of the lower and upper surfaces on the unit mean deviation ρ. In particular, the upper bound on ρ depends critically on the behavior of the roughness correlations of $\delta_1(x,\omega) + \delta_2(x,\omega)$ and $\delta_1(x,\omega) - \delta_2(x,\omega)$. The significance of the estimate (3.8.76) is the fact that as the correlation function of the roughness decreases, the unit mean deviation decreases. Furthermore, as the correlation function

of the roughness increases, the unit mean deviation may increase. In fact, we are currently investigating some of these effects in a systematic way.

b) THE HANGING CABLE PROBLEM

A cable is hung between two pegs. We assume that

(c_1) the cable is perfectly flexible; that it offers no resistance to bending;

(c_2) the cable is not moving any direction and thus no acceleration in any direction;

(c_3) the cable is hanging under its own weight of W units of weight per unit length of cable;

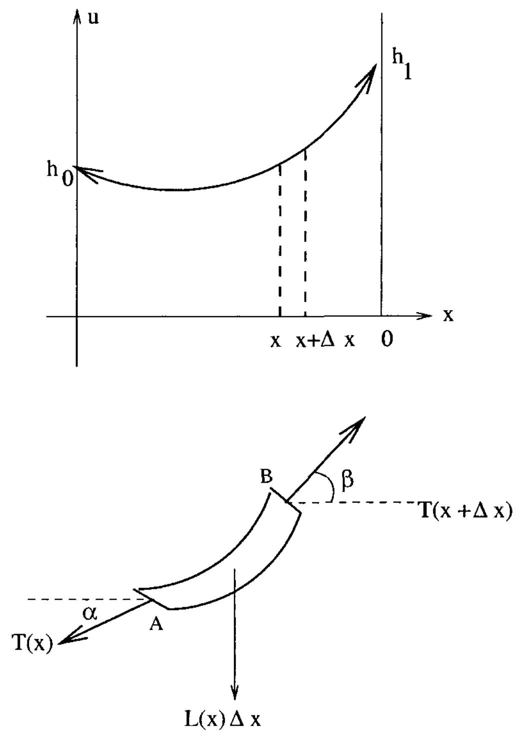
(c_4) W in (c_3) is a random variable. This assumption reflects the change in weight of the cable due to random environmental disturbances such as snow/dust accumulation, etc.

By applying Newton's second law of motion to a segment of the cable between x and $x+\Delta x$, and as a consequence of (c_1), the tensions exerted on the segment AB in Fig. 3.8.3 are tangential to the cable. Let $T(x)$ and $T(x+\Delta x)$ be the forces exerted at the points A and B in the direction of the tangent lines at A and B, respectively. Let $L(x)$ be the loading (force per unit length) on the segment of the cable AB. Therefore $L(x)\Delta x$ will be the total vertical force exerted on the segment of the cable AB of length Δs. This together with (c_3) yields

$$L(x)\Delta x = W\,\frac{\Delta s}{\Delta x}\,\Delta x, \tag{3.8.77}$$

where s represents arc length along the cable. From this and (c_2), the sum of forces in the horizontal and vertical directions are described

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016



Section of cable showing forces acting on it.

Fig. 3.8.3. The hanging cable.

by the following equations

$$\begin{cases} -T(x)\cos\alpha + T(x+\Delta x)\cos\beta = 0 \\ -T(x)\sin\alpha + T(x+\Delta x)\sin\beta - L(x)\Delta x = 0. \end{cases} \tag{3.8.78}$$

By rearranging the first equation in (3.8.78), we obtain,

$$T(x)\cos\alpha = T(x+\Delta x)\cos\beta = T,$$

and assigning T to be the common value, we get

$$T(x) = \frac{T}{\cos\alpha}, \qquad T(x+\Delta x) = \frac{T}{\cos\beta}.$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Substitution of the above expressions into the second equation in (3.8.78) gives us

$$-T\tan\alpha + T\tan\beta = L(x)\Delta x, \tag{3.8.79}$$

where $\tan\alpha$ is the slope of the cable at x and $\tan\beta$ is the slope of the cable at $x + \Delta x$. Let $y(x)$ be the height above the horizontal line. Hence $\tan\alpha = h'(x)$, $\tan\beta = h'(x + \Delta x)$. We note that T can be also a positive random variable. From these considerations and (3.8.79), we have

$$T(\omega)\left[\frac{h'(x+\Delta x,\omega) - h'(x,\omega)}{\Delta x}\right] = L(x).$$

From this (c_4) and (3.8.77), we arrive at

$$T(\omega)\left[\frac{h'(x+\Delta x) - h'(x,\omega)}{\Delta x}\right] = W(\omega)\frac{\Delta s}{\Delta x}. \tag{3.8.80}$$

In the limit as $\Delta x \to 0$

$$\lim_{\Delta x\to 0}\frac{\Delta s}{\Delta x} = \sqrt{1+\left(\frac{dh}{dx}\right)^2}$$

and hence (3.8.80) becomes

$$T(\omega)h''(x,\omega) = W(\omega)\sqrt{1+(h'(x,\omega))^2} \tag{3.8.81}$$

for $x \in (0,1)$ and $h(x)$ must satisfy the boundary conditions

$$h(0,\omega) = h_0(\omega) \qquad \text{and} \qquad h(1,\omega) = h_1(\omega), \tag{3.8.82}$$

where h_0, h_1, $T(\omega)$ and $W(\omega)$ are also random variables defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with means $\widehat{h}_0, \widehat{h}_1, \widehat{T}$, and $\widehat{W}$, respectively.

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

The corresponding smooth boundary value problem in the absence of the random environmental perturbations is given by

$$\widehat{T}m'' = \widehat{W}\sqrt{1+(m')^2}, \qquad m(0) = \widehat{h}_0, \qquad m(1) = \widehat{h}_1. \qquad (3.8.83)$$

The Green's function for (SBVP) (3.8.81)–(3.8.82) and (DBVP) (3.8.83) are given by

$$G(x,s,\omega) = \widehat{G}(x,s) = \begin{cases} (1-x)s & \text{if } 0 \le s \le x \le 1 \\ (1-s)x & \text{if } 0 \le x \le s \le 1. \end{cases} \qquad (3.8.84)$$

We define

$$f(t,x,y,\omega) = \frac{W(\omega)}{T(\omega)}\sqrt{1+y^2}, \qquad \widehat{f}(t,x,y) = \frac{\widehat{W}}{\widehat{T}}\sqrt{1+y^2}.$$

From

$$f(t,x,y,\omega) - f(t,u,v,\omega) = \frac{W(\omega)}{T(\omega)}\left[\int_0^1 \frac{(v+r(y-v))}{\sqrt{1+(v+r(y-v))^2}}\,dr(y-v)\right]$$

we have

$$|f(t,x,y,\omega) - f(t,u,v,\omega)| \le \frac{W(\omega)}{T(\omega)}\,|y-v| \qquad (3.8.85)$$

since $\sup_{r\in R}\left[r/\sqrt{1+r^2}\right] = 1$. Here $k_1(x,\omega) \equiv 0$ and $k_2(s,\omega) = W(\omega)/T(\omega)$

$$\begin{aligned} |f(t,u,v,\omega) - \widehat{f}(t,u,v)| &\le \left|\frac{W(\omega)}{T(\omega)} - \frac{\widehat{W}}{\widehat{T}}\right|\sqrt{1+r^2} \\ &\le \left|\frac{W(\omega)}{T(\omega)} - \frac{\widehat{W}}{\widehat{T}}\right|\left(1+\left(\int_0^1 \frac{rv}{\sqrt{1+r^2v^2}}\,dr\right)v\right) \\ &\le \left|\frac{W(\omega)}{T(\omega)} - \frac{\widehat{W}}{\widehat{T}}\right|(1+|v|). \qquad (3.8.86) \end{aligned}$$

Here

$$k_3(x,\omega) \equiv 0, \qquad k_4(x,\omega) = \left| \frac{W(\omega)}{T(\omega)} - \frac{\widehat{W}}{\widehat{T}} \right| = k_5(x,\omega).$$

Similarly, one can obtain

$$|\widehat{f}(t,u,v)| \leq \frac{\widehat{W}}{\widehat{T}}\,(1+|v|). \tag{3.8.87}$$

Here

$$k_6(t) \equiv 0, \qquad k_7(t) = \frac{\widehat{W}}{\widehat{T}} = k_8(t).$$

Thus $H(\omega)$ is determined by

$$H(\omega)(u,v)(x) = \begin{bmatrix} \frac{W(\omega)}{T(\omega)} \int_0^1 |\widehat{G}(x,s)| v(s)ds \\ \frac{W(\omega)}{T(\omega)} \int_0^1 |\widehat{G}_t(x,s)| v(s)ds \end{bmatrix}. \tag{3.8.88}$$

Here Remark 3.6.1 is applicable. Thus

$$\begin{cases} \psi(x,\omega) = h_0(\omega)(1-x) + h_1(\omega)x, & \widehat{\psi}(x) = \widehat{h}_0(1-x) + \widehat{h}_1 x, \\ \psi'_x(x,\omega) = h_1(\omega) - h_0(\omega) \quad \text{and} & \widehat{\psi}'_x(x) = \widehat{h}_1 - \widehat{h}_0 \end{cases} \tag{3.8.89}$$

$$\begin{cases} M_0(x,\omega) = |\psi(x,\omega) - \widehat{\psi}(t)| \\ \qquad\qquad + \displaystyle\int_0^1 |\widehat{G}(x,s)|(k_4(s,\omega)|m'(s)| + k_5(s,\omega))ds \\ M_1(x,\omega) = |\psi'(x,\omega) - \widehat{\psi}_x(x)| \\ \qquad\qquad + \displaystyle\int_0^1 |\widehat{G}_t(x,s)|(k_4(s,\omega)|m'(s)| + k_5(s,\omega))ds, \end{cases} \tag{3.8.90}$$

where $k_4(x,\omega)$ and $k_5(x,\omega)$ are defined in (3.8.86). We note that the linear operator defined in (3.8.88) has radius if $W(\omega)/2T(\omega) < 1$ w.p. 1. Thus by the application of Theorem 3.6.2, we have

$$(|h(x,\omega) - m(x)|, |h'(x,\omega) - m'(x)|) \leq \sum_{j=0}^{\infty} \left(\frac{W(\omega)}{2T(\omega)}\right)^j (M_0, M_1)(x)$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

for $x \in [0,1]$ which implies

$$|h(x,\omega) - m(x)| \leq \left(1 - \frac{W(\omega)}{2T(\omega)}\right)^{-1} (\|M_0\|_0 + \|M_1\|_0), \qquad (3.8.91)$$

where M_0 and M_1 are defined in (3.8.90). From (3.8.91) one can obtain absolute mean as well as p^{th} mean deviation of solution process of SBVP (3.8.81)–(3.8.82) with its mean DBVP (3.8.83). From (3.8.91), we obtain

$$\begin{aligned} |h(x,\omega) - m(x)| \leq \left(1 - \frac{W(\omega)}{2T(\omega)}\right)^{-1} \Big(& |h_0(\omega) - \widehat{h}_0| + |h_1(\omega) - \widehat{h}_1| \\ & + |h_1(\omega) - \widehat{h}_1 + \widehat{h}_0 - h_0(\omega)| \\ & + \frac{1}{2}\left|\frac{W(\omega)}{T(\omega)} - \frac{\widehat{W}}{\widehat{T}}\right| (\|m'\|_0 + 1)\Big) \end{aligned} \qquad (3.8.92)$$

for $x \in [0,1]$. The estimate for absolute mean deviation is given by

$$E[|h(x,\omega) - m(x)|] \leq \left(\int_\Omega \left(1 - \frac{W(\omega)}{2T(\omega)}\right)^{-2} P(d\omega)\right)^{1/2} \left(\int_\Omega \lambda^2(\omega) P(d\omega)\right)^{1/2}, \qquad (3.8.93)$$

where

$$\begin{aligned} \lambda(\omega) = & |h_0(\omega) - \widehat{h}_0| + |h_1(\omega) - \widehat{h}_1| + |h_1(\omega) - \widehat{h}_1 + \widehat{h}_0 - h_0(\omega)| \\ & + \frac{1}{2}\left|\frac{W(\omega)}{T(\omega)} - \frac{\widehat{W}}{\widehat{T}}\right| (\|m'\|_0 + 1). \end{aligned} \qquad (3.8.94)$$

Furthermore, the relative stability problem can also be investigated by making certain assumptions on the boundary conditions as well as random variable $W(\omega)$ and $T(\omega)$.

3.9. NUMERICAL EXAMPLES

Let $X(\omega)$ be a random variable with uniform distribution on the interval $(0,1)$. We consider numerically the hanging cable problem

$$T(\omega)h''(x,\omega) = W(\omega)\sqrt{1 + h'(x,\omega)^2} \qquad (3.9.1)$$

$$h(0, \omega) = h_0(\omega) = 0.2X(\omega) \tag{3.9.2}$$

$$h(1, \omega) = h_1(\omega) = 0.2X(\omega). \tag{3.9.3}$$

In (3.9.1) $T(\omega) = 1 + 0.1X(\omega)$, $W(\omega) = 1 + 0.1X(\omega)$. We use Euler's method and Runge-Kutta method to discreteize the problem. Below are given tables giving corresponding values of h' and h for various values of X. In the picture, following the tables the broken line graph is that of h while the solid line graph is that of h'. In Table 3.9.1 the results are given by using the Euler method. In Table 3.9.2, the results are based on the Runge-Kutta method. The results concerning the exact soluiton of (3.9.1) are given in Table 3.9.3.

The exact solution of (3.9.1)–(3.9.3) is given by

$$h(x, \omega) = \frac{T(\omega)}{W(\omega)} \cosh\left(K_1 + \frac{W(\omega)}{T(\omega)}\, x\right) + K_2,$$

where

$$K_1 = \sinh^{-1}(\zeta), \qquad K_2 = h_0(\omega) - \frac{T(\omega)}{W(\omega)} \cosh(\sinh^{-1}(\zeta)),$$

and ζ is determined from the equation

$$\left(1 - \cosh\left(\frac{W(\omega)}{T(\omega)}\right)\right)\sqrt{1+\zeta^2} - \left(\sinh\left(\frac{W(\omega)}{T(\omega)}\right)\right)\zeta = \frac{W(\omega)}{T(\omega)}\,(h_0(\omega) - h_1(\omega)).$$

We use (3.8.93) to give a bound for the mean deviation. For (3.9.1) we have the bound 0.27.

Table 3.9.1			Table 3.9.2			Table 3.9.3		
X	h'	h	X	h'	h	X	h'	h
0.0	−0.510	0.095	0.0	−0.510	0.095	0.0	−0.499	0.090
0.1	−0.411	0.054	0.1	−0.411	0.054	0.1	−0.400	0.049
0.2	−0.304	0.019	0.2	−0.304	0.019	0.2	−0.294	0.015
0.3	−0.201	−0.006	0.3	−0.201	−0.006	0.3	−0.191	−0.010
0.4	−0.100	−0.021	0.4	−0.100	−0.021	0.4	−0.090	−0.024
0.5	0.001	−0.025	0.5	0.001	−0.025	0.5	0.010	−0.028
0.6	0.101	−0.019	0.6	0.101	−0.019	0.6	0.110	−0.022
0.7	0.202	−0.004	0.7	0.202	−0.004	0.7	0.212	−0.006
0.8	0.305	0.022	0.8	0.305	0.022	0.8	0.315	0.021
0.9	0.411	0.058	0.9	0.411	0.058	0.9	0.422	0.057
1.0	0.521	0.106	1.0	0.521	0.106	1.0	0.533	0.105

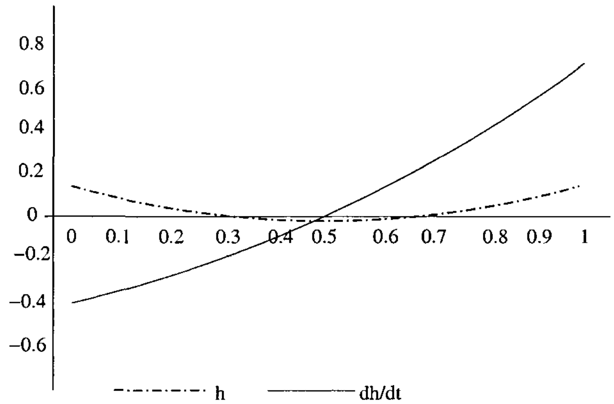


Figure 3.9.1

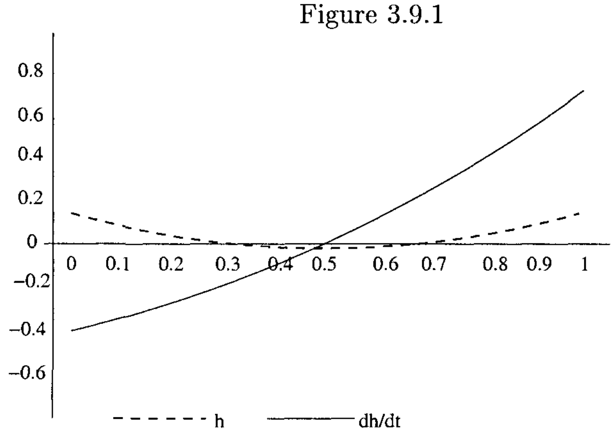


Figure 3.9.2

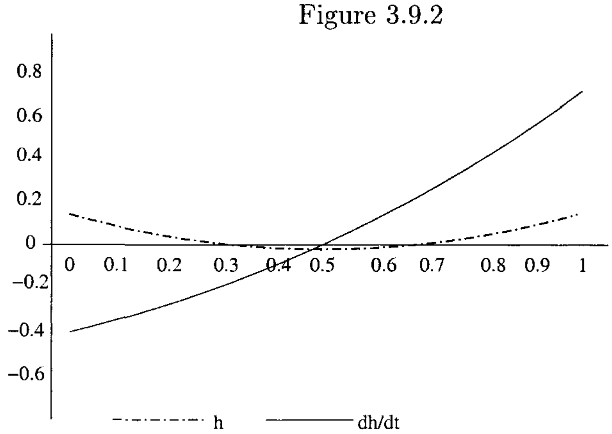


Figure 3.9.3

3.10 NOTES AND COMMENTS

The formulation of boundary value problems with random parameters in the framework of initial value problems with random parameters of Chaptėr 2 is analyzed. The Green's function method is outlined in Section 3.1. The presented results in this section are due to Deimling, Ladde and Lakshmikantham [29] and Chandra, Ladde and Lakshmikantham [17]. Theorems 3.2.1 and 3.2.4 are due to Chandra, Ladde and Lakshmikantham [17]. Theorems 3.2.2 and 3.2.3 are new and are direct extensions of the deterministic results of Chandra, Lakshmikantham and Leela [21]. The contents of Section 3.3 are based on Xia, Boyce and Barry [113], Kozin [53] and Soong [104]. See also Boyce [10], Kohler and Boyce [51], Xia and Boyce [114, 115], and Xia [116, 117]. Theorem 3.4.1 is based on the deterministic result of Chandra, Lakshmikantham and Leela [21], and Bernfeld and Lakshmikantham [4]. See also Ladde and Seikkala [81], Lax [85, 84], and Lax and Boyce [86]. The material of Section 3.5 is based on the deterministic results of Chandra, Lakshmikantham and Leela [21]. The results of Sections 3.6 and 3.7 with regard to Green's function method and comparison method are due to Ladde, Medhin and Sambandham [70] and Chandra, Ladde and Lakshmikantham [17], respectively. The stochastic analysis of compressible gas lubricated slider bearing problem and hanging cable problem are adapted from Chandra, Ladde and Lakshmikhantham [18] and Ladde, Medhin and Sambandham [70], respectively. For related results, see Chow and Saibel [23, 24], Christensen [25], White [111] and Powers [100]. Numerical illustrations and figures in Section 3.9 are taken from Ladde, Medhin and Sambandham [70]. See also Chandra and Ladde [19, 20], Chernov [22], Keller [48, 49], White and Franklin [112].

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

CHAPTER 4: ITÔ-TYPE STOCHASTIC DIFFERENTIAL SYSTEMS

4.0. INTRODUCTION

Under suitable conditions, stochastic mathematical models of dynamic systems described by diffusion processes (probabilistic laws) and systems of stochastic differential equations of Itô-type (Newtonian type laws) are equivalent. The concept of system of stochastic differential equations of Itô-type generates a very natural and straightforward but a difficult problem of "stochastic versus deterministic." In this case, this means that to what extent the solution processes of systems of stochastic differential equations of Itô-type deviate from the solution processes of corresponding systems of deterministic differential equations that are described by drift coefficient vector functions of the corresponding diffusion processes as rate functions. Furthermore, "stochastic versus deterministic" problem can be studied by studying the characterization of the diffusion coefficient square matrix functions of the corresponding diffusion processes. This is because of the fact that the diffusion coefficient matrix function of the diffusion process is a measure of the local magnitude of the random fluctuations of the diffusion process.

A method of generalized variation of constants for a nonlinear systems of Itô-type stochastic differential equations is discussed in Section 4.1. By using a concept of vector Lyapunov like functions and the theory of deterministic system of differential inequalities, several variational comparison theorems are presented in Section 4.2. A method of finding the probability distribution of the solution process of a system of stochastic nonlinear differential equations of Itô-type

is outlined in Section 4.3. Section 4.4 deals with the pth moment stability analysis of the trivial solution of this type of systems. Error estimates and relative stability of Itô-type stochastic differential systems relative to the solutions of the corresponding mean system are analyzed in Sections 4.5 and 4.6, respectively. The mathematical results are applied to the population growth of n-species community model in Section 4.7. Further note that several examples are given to illustrate the scope and the usefulness of the mathematical results.

4.1. VARIATION OF CONSTANTS METHOD

In this section, we devote our discussion to Itô-type stochastic differential system, namely,

$$dy = \widehat{F}(t,y)dt + \sigma(t,y)dz(t), \qquad y(t_0,\omega) = y_0(\omega) \tag{4.1.1}$$

where $\widehat{F}$ is as defined in (2.1.2); $\sigma \in C[R_+ \times R^n, R^{nk}]$ and $z(t) \in R[\Omega, R^k]$ is a normalized Wiener process. It is further assumed that $\widehat{F}$ and σ satisfy desired conditions so that (4.1.1) has solution process. For details, see Ladde and Lakshmikantham [67].

In the following, we formulate theorems similar to Theorems 2.1.1 and 2.1.2 in the context of (2.1.2), (2.1.3) and (4.1.1).

Theorem 4.1.1. *Assume that $\widehat{F}$ and σ in (4.1.1) satisfy desired regularity conditions to assure the existence of solution process $y(t,\omega)$ of (4.1.1) for $t \geq t_0$. Assume that $\widehat{F}(t,x)$ is twice continuously differentiable with respect to x for fixed $t \geq t_0$, and $V \in C[R_+ \times R^n, R^m]$, V_t, V_x and V_{xx} exist and continuous for $(t,x) \in R_+ \times R^n$. Then*

$$\begin{aligned} V(t,y(t,\omega)) &= V(t_0,x(t,\omega)) \\ &\quad + \int_{t_0}^{t} V_x(s,x(t,s,y(s,\omega)))\Phi(t,s,y(s,\omega))\sigma(s,y(s,\omega))dz(s) \end{aligned}$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

$$+\int_{t_0}^{t}[V_s(s,x(t,s,y(s,\omega)))$$
$$+\frac{1}{2}V_x(s,x(t,s,y(s,\omega)))tr\left(\frac{\partial^2 x}{\partial x_0 \partial x_0}(t,s,y(s,\omega))b(s,y(s,\omega))\right)$$
$$+\frac{1}{2}tr\Big(V_{xx}(s,x(t,s,y(s,\omega)))c(t,s,y(s,\omega))\Big)]ds, \quad (4.1.2)$$

where $c(t,s,y) = \Phi(t,s,y)b(s,y)\Phi^T(t,s,y)$, $b(s,y) = \sigma(s,y)\sigma^T(s,y)$, *and* $x(t,\omega) = x(t,t_0,\ y_0(\omega))$ *is the solution process of* (2.1.3).

Proof. Let $y(t,\omega) = y(t,t_0,y_0(\omega))$ and $x(t,\omega) = x(t,t_0,y_0(\omega))$ be solution processes of (4.1.1) and (2.1.3) through $(t_0,y_0(\omega))$, respectively. Under the assumption of $\widehat{F}$, it is known (Theorem A.2.2) that the solution $x(t,t_0,x_0)$ of (2.1.3) is continuously differentiable with respect to t_0 and twice continuously differentiable with respect to x_0. As a result, applying Itô's formula (Theorem A.2.3) to $V(s,x(t,s,y(s,\omega)))$, we have

$$d_sV(s,x(t,s,y(s,\omega))) = V_s(s,x(t,s,y(s,\omega)))ds$$
$$+ V_x(s,x(t,s,y(s,\omega)))d_sx(t,s,y(s,\omega))$$
$$+ \frac{1}{2}tr\Big(V_{xx}(x,s(t,s,y(s,\omega)))d_sx(t,s,y(s,\omega))d_sx^T(t,s,y(s,\omega))\Big).$$

From this and from the fact

$$d_sx(t,s,y(s,\omega)) = \Phi(t,s,y(s,\omega))\sigma(s,y(s,\omega))dz(s)$$
$$+\frac{1}{2}tr\left(\frac{\partial^2 x}{\partial x_0 \partial x_0}(t,s,y(s,\omega))b(s,y(s,\omega))\right)ds, \quad (4.1.3)$$

we obtain

$$d_sV(s,x(t,s,y(s,\omega))) =$$
$$V_x(s,x(t,s,y(s,\omega)))\Phi(t,s,y(s,\omega))\sigma(s,y(s,\omega))dz(s)$$
$$+[V_s(s,x(t,s,y(s,\omega)))$$

$$+\frac{1}{2}V_x(s,x(t,s,y(s,\omega)))tr\left(\frac{\partial^2 x}{\partial x_0 \partial x_0}(t,s,y(s,\omega))b(s,y(s,\omega))\right)$$
$$+\frac{1}{2}tr\Big(V_{xx}(s,x(t,s,y(s,\omega)))\Phi(t,s,y(s,\omega))b(s,y(s,\omega))$$
$$\times\Phi^T(t,s,y(s,\omega))\Big)\Big]ds. \quad (4.1.4)$$

Integrating this form t_0 to t, the desired result (4.1.2) follows.

The following result shows the scope of Theorem 4.1.1.

Corollary 4.1.1. *Let the assumptions of* Theorem 4.1.1 *be satisfied with $n = m$ and $V(t,x) = x$. Then*

$$y(t,\omega) = x(t,\omega) + \int_{t_0}^{t} \Phi(t,s,y(s,\omega))\sigma(s,y(s,\omega))dz(s)$$
$$+\frac{1}{2}\sum_{i,j=1}^{n}\int_{t_0}^{t}\frac{\partial^2 x}{\partial x_{0i}\partial x_{0j}}(t,s,y(s,\omega))b_{ij}(s,y(s,\omega))ds \quad (4.1.5)$$

where $b(s,y) = (b_{ij}(s,y))$ and it is defined in (4.1.2).

We note that this corollary is an analogue of the variation of constants formula for deterministic nonlinear systems of differential equations due to Alekseev [1]. Observe that the requirement of twice continuous differentiability of $\widehat{F}$ with respect to x is due to stochastic nature of the perturbations.

Problem 4.1.1. If $V(t,x) = \|x\|^2$ then (4.1.2) reduces to

$$\|y(t,\omega)\|^2 = \|x(t,\omega)\|^2$$
$$+\int_{t_0}^{t} 2x^T(t,s,y(s,\omega))\Phi(t,s,y(s,\omega))\sigma(s,y(s,\omega))dz(s)$$
$$+\sum_{i,j=1}^{n}\left[\int_{t_0}^{t}\Big(x^T(t,s,y(s,\omega))\frac{\partial^2 x}{\partial x_{0i}\partial x_{0j}}(t,s,y(s,\omega))\right.$$
$$\left.+\sum_{k=1}^{n}\frac{\partial x_k}{\partial x_{0i}}(t,s,y(s,\omega))\frac{\partial x_k}{\partial x_{0j}}(t,s,y(s,\omega))\Big)b_{ij}(s,y(s,\omega))ds\right]. \quad (4.1.6)$$

Example 4.1.1. Consider the differential equation

$$dy = -\frac{1}{2}y^3 dt + \sigma(t,y)dz(t), \qquad y(t_0,\omega) = y_0(\omega) \tag{4.1.7}$$

where $y \in R$ and $z(t)$ is a one-dimensional normalized Wiener process.

$$x' = -\frac{1}{2}x^3, \qquad x(t_0,\omega) = x_0(\omega). \tag{4.1.8}$$

Here $x(t,t_0,x_0) = x_0/[1+(t-t_0)x_0^2]^{1/2}$, $\Phi(t,t_0,x_0) = 1/[1+(t-t_0)x_0^2]^{3/2}$. Let $V(t,x) = x$. Applying Theorem 4.1.1 we have

$$y(t,\omega) = x(t,\omega) + \int_{t_0}^{t} \sigma(s,y(s,\omega))/[1+(t-s)y^2(s,\omega)]^{3/2}dz(s)$$
$$-\frac{1}{2}\int_{t_0}^{t}[3(t-s)y(s,\omega)]/[1+(t-s)y^2(s,\omega)]^{5/2}b(s,y(s,\omega))ds. \tag{4.1.9}$$

where $b(s,y) = \sigma^2(s,y)$.

Remark 4.1.1. We note that for the sake of simplicity, our study is restricted to systems of the form (4.1.1), (2.1.2), and (2.1.3). However, the study is applicable to an arbitrary systems of the following forms:

$$dy = F(t,y,\omega)dt + \Lambda(t,y,\omega)dz, \qquad y(t_0,\omega) = y_0(\omega) \tag{4.1.10}$$

$$m' = \widehat{F}(t,m) \qquad m(t_0) = m_0 = E[y_0(\omega)] \tag{4.1.11}$$

$$x' = \widehat{F}(t,x), \qquad x(t_0,\omega) = x_0(\omega) \tag{4.1.12}$$

where $\widehat{F}$, σ, and $z(t)$ are as defined before; F and column vectors of $\Lambda \in C\big[R_+ \times R^n, R[\Omega, R^n]\big]$ and satisfy required conditions so that (4.1.10) has solution process in the sense of Itô (Ladde and Lakshmikantham [67]). With respect to (4.1.10)–(4.1.12), one can formulate the results analogous to Theorem 4.1.1. We present it as a problem. Details are left to the reader. In fact, the proof can be given by imitating the proofs of Theorems 4.1.1 and 2.1.1.

Problem 4.1.2. Assume that F and Λ in (4.1.10) satisfy desired regularity conditions to assure the existence of solution process of (4.1.10) for $t \geq t_0$. Assume that $\widehat{F}(t,x)$ is twice continuously differentiable with respect to x for fixed $t \geq t_0$, and $V \in C[R_+ \times R^n, R^m]$, V_t, V_x and V_{xx} exist and continuous for $(t,x) \in R_+ \times R^n$. Then

(i) $V(t, y(t,\omega)) = V(t_0, x(t,\omega)) +$
$\int_{t_0}^t V_x(s, x(t,s,y(s,\omega)))\Phi(t,s,y(s,\omega))\Lambda(s,y(s,\omega),\omega)dz(s)$
$+ \int_{t_0}^t L_s V(s, x(t,s,y(s,\omega)))ds$ (4.1.13)

where $x(t) = x(t,t_0,y_0(\omega)) = x(t,\omega)$ and $y(t,\omega) = y(t,t_0,y_0(\omega),\omega)$ are solution processes of (4.1.12) and (4.1.10), respectively; the differential operator L_s is defined by

$$\begin{aligned} L_s V(s, x(t,s,y)) = {} & V_s(s, x(t,s,y)) + V_x(s, x(t,s,y)) \\ & \cdot \left(\frac{1}{2} \operatorname{tr} \left(\frac{\partial^2 x}{\partial x_0 \partial x_0}(t,s,y) B(s,y,\omega) \right) \right. \\ & \left. + \Phi(t,s,y)[F(s,y,\omega) - \widehat{F}(s,y)] \right) \\ & + tr\left(\frac{1}{2} V_{xx}(s, x(t,s,y)) C(t,s,y,\omega) \right), \end{aligned}$$

$B(s,y,\omega) = \Lambda(s,y,\omega)\Lambda^T(s,y,\omega)$ and
$C(t,s,y,\omega) = \Phi(t,s,y)B(s,y,\omega)\Phi^T(t,s,y)$.

(ii) Moreover, if $\widehat{F}(t,x) \equiv 0$, then (4.1.13) reduces to

$$\begin{aligned} V(t, y(t,\omega)) = V(t_0, y_0) & + \int_{t_0}^t V_x(s, y(s,\omega))\Lambda(s, y(\omega), \omega)dz(s) \\ & + \int_{t_0}^t L_s V(s, y(s,\omega))ds \end{aligned} \tag{4.1.14}$$

where

$$\begin{aligned} L_s V(s,y,\omega) = {} & V_s(s,y) + V_x(s,y)F(s,y) \\ & + tr\left(\frac{1}{2} V_{xx}(s,y) B(s,y,\omega) \right). \end{aligned}$$

The following theorem is an analogue of Theorem 2.1.2.

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

Theorem 4.1.2. *Let the hypotheses of* Theorem 4.1.1 *be satisfied. Then*

$$
\begin{aligned}
V(t,y(t,\omega)-\bar{x}(t)) &= V(t_0,x(t,\omega)-\bar{x}(t)) + \int_{t_0}^{t} V_x(s,x(t,s,y(s,\omega)) \\
&\qquad -x(t,s,\bar{x}(s)))\Phi(t,s,y(s,\omega))\sigma(s,y(s,\omega))dz(s) \\
&+\int_{t_0}^{t}\Bigg[V_s(s,x(t,s,y(s,\omega))-x(t,s,\bar{x}(s)))+\frac{1}{2}V_x(s,x(t,s,y(s,\omega)) \\
&\qquad -x(t,s,\bar{x}(s)))tr\left(\frac{\partial^2 x}{\partial x_0 \partial x_0}(t,s,y(s,\omega))\sigma(s,y(s,\omega))\right) \\
&+\frac{1}{2}tr\Big(V_{xx}(s,x(t,s,y(s,\omega))-x(t,s,\bar{x}(s)))c(t,s,y(s,\omega))\Big)\Bigg]ds \qquad (4.1.15)
\end{aligned}
$$

where b and c are as defined in Theorem 4.1.1 *and* $\bar{x}(s) = x(s,t_0,z_0)$ *is the solution of* (2.1.2) *or* (2.1.3) *depending on the choice of* z_0.

Proof. The proof of this theorem can be formulated from the proofs of Theorems 2.1.2 and 4.1.1. We leave the details to the reader.

In the following, we formulate a problem similar to a Problem 2.1.3 in the context of Remark 4.1.1 exhibiting the expression about the deviation of solutions between the problems (4.1.10)–(4.1.12).

Problem 4.1.3. Assume all the conditions of Problem 4.1.2 are valid relative to solution processes of (4.1.10)–(4.1.12). Then

$$
\begin{aligned}
V(t,y(t,\omega)-\bar{x}(t)) &= V(t_0,x(t,t_0,y_0(\omega)-z_0)) \\
&\quad+\int_{t_0}^{t} V_x(s,x(t,s,y(s,\omega)-\bar{x}(s)))\Phi(t,s,y(s,\omega)-\bar{x}(s)) \\
&\qquad \times\Lambda(s,y(s,\omega),\omega)dz(s) \\
&\quad+\int_{t_0}^{t} L_sV(s,x(t,s,y(s,\omega)-\bar{x}(s)),\omega)ds \qquad (4.1.16)
\end{aligned}
$$

where $\bar{x}(t)=\bar{x}(t,\omega)=x(t,t_0,z_0)$ and $x(t,t_0,y_0(\omega)-z_0)$ are solution processes of (4.1.12) through (t_0,z_0) and $(t_0,y_0(\omega)-z_0)$, respectively

and z_0 is either $y_0(\omega)$ or $E[y_0(\omega)] = m_0$; $y(t,\omega)$ is solution process of (4.1.10); L_s is a differential operator defined by

$$\begin{aligned}
L_s V(s, x(t,s,y-\bar{x})) &= V_s(s, x(t,s,y-\bar{x})) \\
&\quad + V_x(s, x(t,s,y-\bar{x}))\left(\frac{1}{2} tr\left(\frac{\partial^2 x}{\partial x_0 \partial x_0}(t,s,y-\bar{x}) B(s,y,\omega)\right)\right. \\
&\quad \left. + \Phi(t,s,y-\bar{x})\Big(F(s,y,\omega) - \widehat{F}(s,\bar{x})\Big)\right) \\
&\quad + tr\Big(\frac{1}{2} V_{xx}(s, x(t,s,y-\bar{x})) C(t,s,y,\omega)\Big),
\end{aligned}$$

B and C are as defined before.

Remark 4.1.2. If we assume $\widehat{F}(t,0) \equiv 0$, then from Lemma A.2.1 system (4.1.1) can be rewritten as

$$dy = \widehat{A}(t,y) y \, dt + \sigma(t,y) dz. \tag{4.1.17}$$

From (2.1.15) and (2.1.13), a remark similar to Remark 2.1.2 can be formulated, analogously.

Example 4.1.2. Let us consider

$$dy = \widehat{A}(t) y \, dt + B(t) y \, dz \tag{4.1.18}$$

where $\widehat{A}(t)$ as defined in (2.1.19), $y \in R^n$, $z(t) \in R$ is a normalized Wiener process. By following Example 2.1.1 and using (4.1.1) in the context of (4.1.18), (2.1.19) and (2.1.20), expressions similar to (2.1.21) and (2.1.22) are given by

$$\begin{aligned}
\|y(t,\omega)\|^2 &= \|x(t,\omega)\|^2 \\
&\quad + 2\int_{t_0}^{t} y^T(s,\omega)\Phi^T(t,s)\Phi(t,s)B(s)y(s,\omega)dz(s) \\
&\quad + \int_{t_0}^{t} tr(\Phi(t,s)B(s)y(s,\omega)y^T(s,\omega)B^T(s)\Phi(t,s))ds,
\end{aligned} \tag{4.1.19}$$

and

$$\begin{aligned}\|y(t,\omega)-\bar{x}(t)\|^2 &= \|x(t,\omega)-\bar{x}(t)\|^2 \\ &+\int_{t_0}^{t} 2(y(s,\omega)-\bar{x}(s))^T\Phi^T(t,s)\Phi(t,s)B(s)y(s,\omega)dz(s) \\ &+\int_{t_0}^{t} tr(\Phi(t,s)B(s)y(s,\omega)y^T(s,\omega)B^T(s)\Phi^T(t,s))ds, \quad (4.1.20)\end{aligned}$$

respectively.

Problem 4.1.4. If $V(t,x)=\|x\|^2$, then equation (4.1.15) in Theorem 4.1.2 becomes

$$\begin{aligned}&\|y(t,\omega)-\bar{x}(t)\|^2 = \|x(t,\omega)-\bar{x}(t)\|^2 \\ &+2\int_{t_0}^{t}(x(t,s,y(s,\omega))-x(t,s,\bar{x}(s)))^T\Phi(t,s,y(s,\omega))\sigma(s,y(s,\omega))dz(s) \\ &+\sum_{i,j=1}^{n}\Bigg[\int_{t_0}^{t}(x(t,s,y(s,\omega))-x(t,s,\bar{x}(s)))^T\frac{\partial^2 x}{\partial x_{0i}\partial x_{0i}}(t,s,y(s,\omega)) \\ &+\sum_{k=1}^{n}\frac{\partial x_k}{\partial x_{0i}}(t,s,y(s,\omega))\frac{\partial x_k}{\partial x_{0j}}(t,s,y(s,\omega))b_{ij}(t,s,y(s,\omega))ds\Bigg]. \quad (4.1.21)\end{aligned}$$

Problem 4.1.5. Derive an expression analogous to the expression in Problem 4.1.3 for the solution processes of (4.1.18) in the context of Problem 4.1.4.

4.2. COMPARISON METHOD

In this section, we develop a few comparison theorems relative to the system of stochastic differential equations of Itô-type. These comparison theorems are obtained by using the concept of vector Lyapunov-like functions and the theory of deterministic differential inequalities.

One of the fundamental comparison theorems is as follows.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Theorem 4.2.1. *Assume that*

(i) *all the hypotheses of* Theorem 4.1.1 *are satisfied;*

(ii) $V(s, x(t, s, y))$ *satisfies*

$$L_s V(s, x(t,s,y)) \leq g(s, V(s, x(t, s, y(s)))), \tag{4.2.1}$$

where

$$\begin{aligned} L_s V(s, x(t,s,y)) &= V_s(s, x(t,s,y)) \\ &+ V_x(s, x(t,s,y)) tr\left(\frac{1}{2}\frac{\partial^2 x}{\partial x_0 \partial x_0}(t,s,y)b(s,y)\right) \\ &+ tr\left(\frac{1}{2}V_{xx}(s, x(t,s,y))c(t,s,y)\right), \end{aligned} \tag{4.2.2}$$

$b(s,y) = \sigma(s,y)\sigma^T(s,y)$ *and* $c(t,s,y) = \Phi(t,s,y)b(s,y)\phi^T(t,s,y)$*;*

(iii) $g \in C[R_+ \times R^m, R^m]$, $g(t,u)$ *is concave and quasi-monotone non-decreasing in* u *for each* $t \in R_+$, *and* $r(t) = r(t, t_0, u_0)$ *is the maximal solution of*

$$u' = g(t,u), \qquad u(t_0) = u_0 \tag{4.2.3}$$

existing for $t \geq t_0$*;*

(iv) $y(t,\omega) = y(t, t_0, y_0(\omega), \omega)$ *and* $x(t,\omega) = x(t, t_0, y_0(\omega))$ *with* $x_0(\omega) = y_0(\omega)$ *are solution processes of* (4.1.1) *and* (2.1.3), *respectively, and* $E[V(s, x(t, s, y(s,\omega)))]$ *exists for* $t \geq s \geq t_0$, *where* $x(t, s, y(s,\omega))$ *is the solution process of* (2.1.3) *through* $(s, y(s,\omega))$.

Then

$$E[V(t, y(t,\omega))] \leq r(t, t_0, u_0), \qquad t \geq t_0 \tag{4.2.4}$$

whenever

$$E[V(t_0, x(t,\omega))] \leq u_0. \tag{4.2.5}$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

Proof. Set

$$w(s) = E[V(s, x(t, s, y(s, \omega)))], \qquad w(t_0) = E[V(t_0, x(t, \omega))].$$

Then assumption (iv), together with the regularity conditions on $V(s, x(t, s, y(s, \omega))$ and $y(s, \omega)$, implies that $w(s)$ is continuous for $t \geq s \geq t_0$. Applying Itô's formula to $V(s, x(t, s, y(s, \omega)))$ with respect to s, we get

$$\begin{aligned}
d_s V(s, x(t, s, y(s, \omega))) = \\
&V_x(s, x(t, s, y(s, \omega)))\Phi(t, s, y(s, \omega))\sigma(s, y(s, \omega))dz(s) \\
&+V_s(s, x(t, s, y(s, \omega))) \\
&+\frac{1}{2}V_x(s, x(t, s, y(s, \omega)))tr\left(\frac{\partial^2 x}{\partial x_0 \partial x_0}(t, s, y(s, \omega))b(s, y(s, \omega))\right) \\
&+\frac{1}{2}\, tr\Big(V_{xx}(s, x(t, s, y(s, \omega)))\Phi(t, s, y(s))b(s, y(s, \omega))\Phi^T(t, s, y(s, \omega))\Big)ds.
\end{aligned}$$

For $h > 0$ sufficiently small such that $t \geq s + h$, this, together with assumption (ii) and concavity of $g(t, u)$ implies

$$\begin{aligned}
E[V(s+h, x(t, s+h, y(s+h, \omega)))] - E[V(s, x(t, s, y(s, \omega)))] \\
&= E\left[\int_s^{s+h} L_s V(u, x(t, u, y(u, \omega)))du\right] \\
&\leq E\left[\int_s^{s+h} g(u, V(u, x(t, u, y(u, \omega))))\right] du \\
&\leq \int_s^{s+h} g(u, E[V(u, x(t, u, y(u, \omega)))])du. \qquad (4.2.6)
\end{aligned}$$

It therefore follows that

$$D^+ w(s) \leq g(s, w(s)), \qquad t > s \geq t_0.$$

An application of Theorem A.2.4 yields immediately the stated result (4.2.4). This completes the proof of the theorem.

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

A variation of comparison Theorem 4.2.1 in the framework of Remark 4.1.1 and Problem 4.1.2 is formulated as a corollary whose proof can be formulated analogously.

Corollary 4.2.1. *Assume that all hypotheses of* Theorem 4.2.1 *except* (i) *are satisfied. Hypothesis* (i) *and* L_s *in* (4.2.2) *are replaced by the conditions and* L_s *described in* Problem 4.1.2. *Then,* (i) *the conclusion of the* Theorem 4.2.1 *remains valid.* (ii) *Moreover, if* $\widehat{F}(t,0) \equiv 0$, *then the corollary reduces to the usual comparison theorem* (Ladde [58]) *as a special case.*

Remark 4.2.1. The drawback of Theorem 4.2.1 is assumption (iv). However, under certain conditions one could show that assumption (iv) holds. For example, let $V(t,x) \geq 0$ and $V(t,x) \leq a(t, \|x\|)$ where $a \in C[R_+ \times R_+, R_+^m]$ and $a(t,u)$ is concave in u for fixed $t \in R_+$. Then we could have

$$0 \leq E[V(s, x(t,s,y(s,\omega)))] \leq a\Big(s, E[\|x(t,s,y(s,\omega))\|]\Big)$$

from which (iv) follows using concavity of a and properties of solution processes of $x(t,s,z)$ and $y(t,\omega)$.

The examples given below illustrate the nature and the scope of Theorem 4.2.1.

Example 4.2.1. Consider the scalar stochastic differential equation (4.1.7) and its mean (4.1.8). Further assume that $\|y_0\|_2 < \infty$ and

$$\sigma^2(t,y) \leq \lambda(t)|y|^2 \qquad \text{for } (t,y) \in R_+ \times R, \tag{4.2.7}$$

where $\lambda \in C[R_+, R_+]$. As before, we recall that

$$x(t,t_0,x_0) = x_0/[1 + (t-t_0)x_0^2]^{1/2},$$
$$\Phi(t,t_0,x_0) = 1/[1 + (t-t_0)x_0^2]^{3/2}.$$

Furthermore,

$$\frac{\partial^2 x}{\partial x_0 \partial x_0}(t, t_0, x_0) = -3(t-t_0)x_0/[1+(t-t_0)x_0^2]^{5/2}.$$

By taking $V(t,x) = \frac{1}{2}|x|^2$, we compute $L_sV(s, x(t,s,y))$ relative to (4.1.7) as

$$\begin{aligned} L_sV(s,x(t,s,y)) &= \frac{1}{2}x(t,s,y)\frac{\partial^2 x}{\partial x_0 \partial x_0}(t,t_0,x_0)\sigma^2(s,y) \\ &\quad + \frac{1}{2}\left(\frac{\partial x}{\partial x_0}(t,s,y)\sigma(s,y)\right)^2 \\ &= \frac{1}{2}(-3(t-s)y^2+1)\sigma^2(s,y)/[1+(t-s)y^2]^3. \end{aligned}$$

This together with (4.2.7), the nature of underline functions and $V(t,x) = \frac{1}{2}|x|^2$, yields

$$L_sV(s,x(t,s,y)) \leq \lambda(t)V(s,x(t,s,y))$$

for $t_0 \leq s \leq t$ and $y \in R$. In this case, the comparison equation (4.2.3) reduces to

$$u' = \lambda(t)u, \qquad u(t_0) = u_0 \tag{4.2.8}$$

which satisfies the hypothesis of Theorem 4.2.1 with

$$r(t,t_0,u_0) = u_0 \exp\left[\int_{t_0}^{t} \lambda(\nu)d\nu\right]$$

and $E[V(t_0, x(t,t_0,y_0))] \leq u_0$. We further note that $E[V(s,x(t,s,y(s,\omega)))]$ exists. Hence, we apply Theorem 4.2.1 and conclude that

$$\frac{1}{2}E[|y(t,\omega)|^2] \leq u_0 \exp\left[\int_{t_0}^{t} \lambda(s)ds\right], \qquad t \geq t_0. \tag{4.2.9}$$

In particular, if $E[V(t_0,x(t,t_0,y_0))] = u_0 = \frac{1}{2}E[|x(t,t_0,y_0)|^2]$ then (4.2.9) becomes

$$E[|y(t,\omega)|^2] \leq E[|x(t,\omega)|^2] \exp\left[\int_{t_0}^{t} \lambda(s)ds\right], \qquad t \geq t_0. \tag{4.2.10}$$

where $x(t,\omega) = x(t,t_0,y_0)$.

Example 4.2.2. Let us consider the system (4.1.18) and its average system (2.1.20) with initial data $y(t_0, \omega) = y_0 = x(t_0, \omega)$. By setting $V(t, x) = \frac{1}{2}\|x\|^2$, we compute $L_s V(s, x(t, s, y))$ with respect to (4.1.18).

$$L_s V(s, x(t, s, y)) = \frac{1}{2} tr(\Phi(t, s)B(s)yy^T B^T(s)\Phi^T(t, s)). \quad (4.2.11)$$

By assuming

$$tr(\Phi(t, s)B(s)yy^T B^T(s)\Phi^T(t, s)) \leq \lambda(t)\|\Phi(t, s)y\|^2 \quad (4.2.12)$$

where $\lambda \in C[R_+, R_+]$, (4.2.11) can be written as

$$L_s V(s, x(t, s, y)) \leq \lambda(t)V(s, x(t, s, y)).$$

Again the comparison equation is as in (4.2.8). By following the rest of the discussion similar to the Example 4.2.1, we have

$$E[\|y(t, \omega)\|^2] \leq E[\|x(t, \omega)\|^2] \exp\left[\int_{t_0}^{t} \lambda(s)ds\right] \quad (4.2.13)$$

where $y(t, \omega)$ and $x(t, \omega)$ are the solution processes of (4.1.18) and (2.1.20) through (t_0, y_0), respectively.

Problem 4.2.1. Consider the following differential equation

$$dy = \alpha y\, dt + \beta y dz, \qquad y(t_0, \omega) = y_0$$

where $\alpha, \beta \in R$ and z is a normalized Wiener process. Show that

$$E[|y(t, \omega)|^2] \leq E[|x(t, \omega)|^2] \exp[\beta^2(t - t_0)],$$

where $y(t, \omega)$ and $x(t, \omega)$ are the solution processes of the above stochastic differential equation and

$$\dot{x} = \alpha x, \qquad x(t_0, \omega) = y_0,$$

respectively.

Finally, a theorem similar to Theorem 2.2.2 with regard to (4.1.1), (2.1.2) and (2.1.3) is formulated as follows.

Theorem 4.2.2. *Let the hypotheses of* Theorem 4.2.1 *be satisfied except* (4.2.1) *replaced by*

$$L_sV(s, x(t, s, y) - x(t, s, z)) \le g(s, V(s, x(t, s, y) - x(t, s, z))), \tag{4.2.14}$$

where L_s is as defined in (4.2.1). *Then*

$$E[V(t, y(t, \omega) - x(t))] \le r(t, t_0, u_0), \qquad t \ge t_0 \tag{4.2.15}$$

whenever

$$E[V(t_0, x(t, \omega) - \bar{x}(t))] \le u_0.$$

Proof. The proof of theorem follows from the proofs of Theorems 2.2.2 and 4.2.1.

Remark 4.2.2. Another variation of comparison Theorem 4.2.2 in the context of Remark 4.1.1, Corollary 2.2.2 and Problem 4.1.3 can be reformualted. The details are left to the reader.

Example 4.2.3. Consider the system (4.1.18) and its mean systems (2.1.19) and (2.1.20). Let $y(t)$ be the solution process of (4.1.18) and $\bar{x}(t) = x(t, t_0, z_0)$ be the solution process of either (2.1.19) or (2.1.20) with $x_0 = y_0$ depending on the value of z_0. Let $V(t, x) = \frac{1}{2}|x|^2$. By following Theorem 4.2.2 and Example 4.2.2, we have

$$L_sV(s, x(t, s, y) - x(t, s, z)) = \frac{1}{2}tr(\Phi(t, s)B(s)yy^TB^T(s)\phi^T(t, s))$$

which can be rewritten as

$$L_sV(s, x(t, s, y) - x(t, s, z)) \le \\ \lambda(s)(V(s, x(t, s, y) - x(t, s, z)) + V(s, x(t, s, z))) \tag{4.2.16}$$

provided Φ, B satisfies the relation (4.2.12). We note that we have used the fact that $tr(A+C) = tr(A)+tr(C)$. The comparison equation is

$$u' = \lambda(s)(u + V(s, x(t, s, z))), \qquad u(t_0) = u_0. \tag{4.2.17}$$

By using the argument in the preceding example, we arrive at

$$E[\|y(t,\omega) - \bar{x}(t)\|^2] \leq E[\|x(t,\omega) - \bar{x}(t)\|^2] \exp\left[\int_{t_0}^{t} \lambda(s)ds\right] + \int_{t_0}^{t} \lambda(s)V(s, x(t, s, x(s))) \exp\left[\int_{s}^{t} \lambda(r)dr\right] ds. \tag{4.2.18}$$

4.3. PROBABILITY DISTRIBUTION METHOD

In this section, we discuss the method of finding the probability distribution of the solution process of the system of stochastic differential equations of Itô-type

$$dy = \widehat{F}(t, y)dt + \sigma(t, y)dz(t), \qquad y(t_0, \omega) = y_0(\omega) \tag{4.3.1}$$

where $\widehat{F}$, and $z(t)$ are as defined in (4.1.1). The transition probability density function of the diffusion process corresponding to the solution process of (4.3.1) is determined by the following theorem.

Theorem 4.3.1. *Let $y(t)$ be an n-dimensional diffusion process determined by (4.3.1). Assume that $y(t)$ possesses a transition probability density function $p(y,t) \equiv p(y, t, y_0, t_0)$. Further assume that $\frac{\partial p}{\partial t}$, $\frac{\partial(\widehat{F}(t,y)p(y,t))}{\partial y}$ and $\frac{\partial^2(b(t,y)p(y,t))}{\partial y \partial y}$ exists and are continuous functions and $b(t, y) = \sigma(t, y)\sigma^T(t, y)$. Then for fixed (t_0, y_0) such that $t_0 \leq t$, this transition probability density function $p(y, t)$ is the solution of the Kolmogorov forward equation or the Fokker-Plank equation*

$$\frac{\partial p}{\partial t} + \sum_{j=1}^{n} \frac{\partial(\widehat{F}_j(t,y)p(y,t))}{\partial y_j} - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2(b_{jk}(t,y)p(y,t))}{\partial y_j \partial y_k} = 0$$

$$p(y, t_0, y_0, t_0) = \prod_{j=1}^{n} \delta(y_j - y_{0j}). \tag{4.3.2}$$

Proof. The proof of this theorem can be found in any standard book on stochastic processes.

Remark 4.3.1. The choice of a method of solution of the Fokker-Plank equation with the initial and boundary conditions depends largely on the form of $\widehat{F}(t, y)$. If the functions $\widehat{F}(t, y)$ and $\sigma(t, y)$ are nonlinear then a general solution of (4.3.2) is difficult to find. If $\widehat{F}(t, y)$ is linear and time-invariant and if $\sigma(t, y)$ is constant matrix, then the solution of (4.3.2) can be obtained by the method of variables separable or the method of Fourier transform.

Example 4.3.1. We consider the Itô-type system of stochastic differential equations

$$dy = Ay\,dt + \sigma dz, \qquad y(t_0) = y_0 \tag{4.3.3}$$

where $z(t)$ is n-dimensional normalized Wiener process;
$A = dig(a_1, a_2, \ldots, a_n)$ and σ are $n \times n$ constant matrices. The Fokker-Plank equation relative to (4.3.3) has the following form

$$\frac{\partial p}{\partial t} + \sum_{j=1}^{n} a_j \frac{\partial(y_j p)}{\partial y_j} - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} b_{jk} \frac{\partial^2 p}{\partial y_j \partial y_k} = 0 \tag{4.3.4}$$

$$p(y, t_0, y_0, t_0) = \prod_{j=1}^{n} \delta(y_j - y_{0j})$$

$$p(y, t, y_0, t_0) \to 0 \qquad \text{as } y_j \to \pm\infty, \qquad j = 1, 2, \ldots, n.$$

We employ the Fourier transform method to solve (4.3.4). Let us consider the characteristic function $\phi(u, t)$ which is the n-dimensional

Fourier transform of $p(y,t) \equiv p(y,t,y_0,t_0)$, that is,

$$\phi(u,t) = \mathcal{F}[p(y,t,y_0,t_0)] = \int_{-\infty}^{\infty} \exp\left[i\sum_{j=1}^{n} u_j y_j\right] p(y,t)dy.$$

From the properties of Fourier transform, we have

$$\mathcal{F}\left[\frac{\partial p}{\partial t}\right] = \frac{\partial \phi}{\partial t}, \qquad \mathcal{F}\left[\frac{\partial (y_j p(y,t))}{\partial y_j}\right] = -u_j\frac{\partial \phi}{\partial u_j},$$
$$\mathcal{F}\left[\frac{\partial^2 p}{\partial y_j y_k}\right] = -u_j u_k \phi. \tag{4.3.5}$$

Hence, by taking the Fourier transform of the partial differential equation (4.3.4) with the initial condition, we obtain the following first order partial differential equation

$$\frac{\partial \phi}{\partial t} - \sum_{j=1}^{n} a_j u_j \frac{\partial \phi}{\partial u_j} + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} b_{ij} u_j u_k \phi = 0. \tag{4.3.6}$$

This equation (4.3.6) is connected with Liouville equation in Theorem 2.3.1. Therefore, the Lagrange system corresponding to (4.3.6) reduces to

$$\frac{du_j}{dt} = -a_j u_j, \qquad j = 1, 2, \ldots, n$$
$$\frac{d\phi}{dt} = -\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{2}\, b_{jk} u_j u_k\right)\phi. \tag{4.3.7}$$

By solving this system of ordinary differential equations, we obtain

$$u_j = c_j \exp[-a_j t], \qquad j = 1, 2, \ldots, n$$

$$\phi = c_{n+1} \exp\left[\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{2} b_{jk} u_j u_k (a_j + a_k)^{-1}\right], \tag{4.3.8}$$

where c_j for $j = 1, 2, \ldots, n+1$ are arbitrary constants of integration. Therefore, the general solution of (4.3.6) has the following form

$$\phi(u,t) = \psi(u_1 \exp[a_1 t], u_2 \exp[a_2 t], \ldots, \\ u_n \exp[a_n t]) \exp\left[\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{1}{2} b_{jk} u_j u_j (a_j + a_k)^{-1}\right],$$

where ψ is an arbitrary function to be determined by using the initial condition in (4.3.4). From this discussion, we have

$$\begin{aligned}\psi(u_1, u_2, \ldots, u_n) &= \phi(u, t_0) \exp\left[-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} b_{jk} u_j u_k (a_k + a_j)^{-1}\right] \\ &= \exp\left[i \sum_{j=1}^{n} u_j u_{0j} - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} b_{jk} u_j u_k (a_j + a_k)^{-1}\right]\end{aligned}$$

and hence

$$\begin{aligned}&\phi(u,t) = \\ &\exp\left[i \sum_{j=1}^{n} u_j y_{0j} \exp[a_j t] - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} b_{jk} u_j u_k (a_j + a_k)^{-1} \exp[(a_j + a_k)t]\right] \\ &\qquad \times \exp\left[\sum_{j=1}^{n}\sum_{k=1}^{n} b_{jk} u_j u_k (a_j + a_k)^{-1}\right] \\ &= \exp\left[i \sum_{j=1}^{n} u_j y_{0j} \exp[a_j t] \right. \\ &\qquad \left. + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} b_{jk} u_j u_k (a_j + a_k)^{-1} (1 - \exp[(a_j + a_k)t])\right]. \quad (4.3.9)\end{aligned}$$

The solution corresponding to this characteristic function clearly defines a n-dimensional Gaussian distribution. Thus, the solution of $p(y,t)$ of (4.3.4) has the form

$$p(y, t, y_0, t_0) = (2\pi)^{-n/2} |\Lambda|^{-1/2} \exp\left[-\frac{1}{2}(y - m)^T \Lambda^{-1} (y - m)\right], \quad (4.3.10)$$

where the components of the mean vector m and the covariance matrix Λ are

$$m_j(t) = y_{oj}\exp[a_j t], \qquad \Lambda_{jk} = -\frac{b}{(a_j + a_k)}[1 - \exp[(a_j + a_k)t]]$$

for $j, k = 1, 2, \ldots, n$.

Remark 4.3.2. A remark similar to Remark 2.3.3 can be formulated, analogously.

4.4. STABILITY ANALYSIS

The following results provide sufficient conditions for the p-th moment stability properties of the trivial solution process of (4.1.1). In the following discussion, we assume that (4.1.1) has p-th order solution process.

Theorem 4.4.1. *Assume that all the hypotheses of* Theorem 4.2.1 *are satisfied. Further suppose that* $\widehat{F}(t,0) \equiv 0$, $\sigma(t,0) \equiv 0$ *and* $g(t,0) \equiv 0$, *and for all* $(t,x) \in R_+ \times R^n$,

$$b(\|x\|^p) \le \sum_{i=1}^{m} V_i(t,x) \le a(t, \|x\|^p) \tag{4.4.1}$$

where $b \in \mathcal{VK}$, $a \in \mathcal{CK}$, *and* $p \ge 1$. *Then*

(JM$_1$) *of* (4.2.3) *and* (2.1.3) *implies* (SM$_1$) *of* (4.1.1), *and*

(JM$_2$) *of* (4.2.3) *and* (2.1.3) *implies* (SM$_2$) *of* (4.1.1).

Proof. The proof of theorem can be reformulated by following the proof of Theorem 2.4.1 in the context of Theorem 4.2.1 and Remark 2.2.1. The details are left to the reader as an exercise.

Now we present some examples to justify the importance of Theorem 4.4.1 and the concept of joint stability.

Example 4.4.1. Let us consider Example 4.2.1. We assume that $\sigma(t, 0) \equiv 0$, and λ in (4.2.7) belongs to $L^1[R_+, R_+]$, that is,

$$\int_0^\infty \lambda(s)ds < \infty. \tag{4.4.2}$$

It is obvious that

$$\nu(t, t_0) = \frac{1}{2}E[\|x(t, t_0, y_0(\omega))\|^2] \exp\left[\int_{t_0}^t \lambda(s)ds\right]. \tag{4.4.3}$$

This together with the assumptions about σ and λ, it follows that the trivial solution processes of (4.2.8) and (4.1.8) are jointly stable in the mean. From this and the application of Theorem 4.4.1, one can conclude that the trivial solution of (4.1.7) is stable in the mean-square sense. Moreover, it is asymptotically stable in the mean-square sense, since $\nu(t, t_0)$ in (4.4.3) tends to zero as $t \to \infty$ as long as the second moment of y_0 is finite.

Example 4.4.2. Let us consider Example 4.2.2. We assume that λ in (4.2.12) satisfies (4.4.2) and $\|y_0\|_2 < \infty$. From the discussion of Example 4.4.1 and inequality (4.2.13), it is obvious that the trivial solution of (4.1.18) is mean-square stable and asymptotically stable, whenever the trivial solution of (2.1.20) is stable and asymptotically stable.

Theorems analogous to Theorems 2.4.2 and 2.4.3 for Itô-type systems of differential equations (4.1.1) will be the subject of the remaining part of this section. Throughout our discussion, we assume that (4.1.1) possesses p-th order solution process.

Theorem 4.4.2. *Let the hypotheses of* Theorem 4.1.1 *be satisfied. Assume that* V, $x(t, \omega)$, $\Phi(t, s, y(s, \omega))$, $\sigma(s, y(s, \omega))$, *and* $\widehat{F}(t, y)$ *satisfy the following conditions*

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

(i) $b(\|x\|^p) \le \sum_{i=1}^{m} |V_i(t,x)| \le a(\|x\|^p)$ *for all* $(t,x) \in R_+ \times R^n$ *where* $p \ge 1$, $b \in \mathcal{VK}$ *and* $a \in \mathcal{CK}$;

(ii) $\widehat{F}(t,0) \equiv 0$ *and* $\sigma(t,0) \equiv 0$ *for* $t \in R_+$;

(iii) $E[V_x(s, x(t,s,y(s,\omega)))\Phi(t,s,y(s,\omega))\sigma(t,s,y(s,\omega))]$ *exists for* $t \ge t_0$;

(iv) $\sum_{i=1}^{m} |L_s V_i(s, x(t,s,y))| \le \lambda(s) \sum_{i=1}^{m} |V_i(s,y)|$ *for* $t_0 \le s \le t$ *and* $\|y\|^p \le \rho$, *where*

$$L_s V(s, x(t,s,y)) = V_s(s, x(t,s,y)) + V_x(s, x(t,s,y)) tr\left(\frac{1}{2}\frac{\partial^2 x}{\partial x_0 \partial x_0}(t,s,y) b(s,y)\right) + tr\left(\frac{1}{2} V_{xx}(s, x(t,s,y)) c(t,s,y)\right),$$

$$c(t,s,y) = \Phi(t,s,y) b(s,y) \Phi^T(t,s,y),$$

$$b(s,y) = \sigma(s,y)\sigma^T(s,y),$$

and $x(t,s,y)$ *is the solution process of* (2.1.3) *through* (s,y), $\rho > 0$ *and* $\lambda \in C[R_+, R_+] \cap L^1(R_+, R_+]$;

(v) $E\left[\sum_{i=1}^{m} V_i(t_0, x(t,\omega))\right] \le \alpha(E[\|y_0(\omega)\|^p])$, *whenever* $E[\|y_0(\omega)\|^p] \le \rho$ *for some* $\rho > 0$, *where* $\alpha \in \mathcal{CK}$. *Then the trivial solution process of* (4.1.1) *is stable in the p-th mean.*

Proof. Let $y(t,\omega)$ be a solution process of (4.1.1) and let $x(t,s,y(s,\omega))$ and $x(t,\omega) = x(t,t_0,y_0(\omega))$ be the solution process of (2.1.3) through $(s, y(s,\omega))$ and $(t_0, y_0(\omega))$, respectively, for $t_0 \le s \le t$ and $t_0 \in R_+$. From hypothesis (ii) $x(t,t_0,0) \equiv 0$ and $y(t,t_0,0) \equiv 0$ be the trivial solution process of (2.1.3) and (4.1.1), respectively. From (4.1.2) and assumption (iii), we obtain

$$E[|V_i(t, y(t,\omega))|] \le E[|V_i(t_0, x(t,\omega))|] + \int_{t_0}^{t} E[|L_s V_i(s, x(t,s,y(s,\omega)))|] ds.$$

This together with hypotheses (iv) and (v), yields

$$m(t) \leq \alpha(E[\|y_0(\omega)\|^p]) + \int_{t_0}^{t} \lambda(s)m(s)ds, \tag{4.4.4}$$

as long as $E[\|y(s,\omega)\|^p] \leq \rho$, where $m(t) = \sum_{i=1}^{m} E[|V_i(t, y(t,\omega))|]$. By applying Lemma A.2.4, we get

$$m(t) \leq \alpha(E[\|y_0(\omega)\|^p]) \exp\left[\int_{t_0}^{t} \lambda(s)ds\right],$$

as long as $E[\|y(s,\omega)\|^p] \leq \rho$ for $t_0 \leq s \leq t$. This implies

$$\sum_{i=1}^{m} E[|V_i(t, y(t,\omega))|] \leq \alpha(E[\|y_0(\omega)\|^p]) \exp\left[\int_{t_0}^{t} \lambda(s)ds\right], \tag{4.4.5}$$

as long as $E[\|y(s,\omega)\|^p] \leq \rho$ for $t_0 \leq s \leq t$. First we show that the above inequality is valid for all $t \geq t_0$. For this purpose, we choose $y_0(\omega)$ such that

$$K\alpha(E[\|y_0(\omega)\|^p]) < b(\rho), \tag{4.4.6}$$

where $K = \exp\left[\int_{t_0}^{\infty} \lambda(s)ds\right]$. We claim that $E[\|y(t,\omega)\|^p] < \rho$ for all $t \geq t_0$ whenever (4.4.6) holds. Assume this claim is false, then there exist $\bar{t}$ and $y_0(\omega)$ such that $\bar{t} > t_0$, $E[\|y_0(\omega)\|^p] < \rho$, $E[\|y(t,\omega)\|^p] < \rho$ for $t_0 \leq t < \bar{t}$ and $E[\|y(\bar{t},\omega)\|^p] = \rho$. From this (4.4.5) is valid for $t \in [t_0, \bar{t}]$ and hence using hypotheses (i) and (4.4.6), we get

$$b(\rho) = b(E[\|y(\bar{t},\omega)\|^p]) \leq E[b(\|y(\bar{t},\omega)\|^p] \leq \sum_{i=1}^{m} E[V_i(\bar{t}, y(\bar{t},\omega))] < b(\rho).$$

This contradiction establishes the impossibility of the existence of such a $\bar{t}$. This verifies the validity of the claim. Hence (4.4.5) is true for all $t \geq t_0$. From this fact, inequality (4.4.2) and the argument used in the final part of the proof of Theorem 2.4.2, the conclusion of the theorem follows immediately. This completes the proof.

Remark 4.4.1. We remark that one may question the feasibility of hypothesis (iii) of Theorem 4.4.2. We observe that Lemma A.2.2 and hypothesis (v) provides a suitable reason to believe the validity of hypothesis (iii). Of course, the feasibility of hypothesis (v) is justified in Remark 2.4.6. Furthermore, we recall that these are sufficient conditions for the p-th moment stability.

The next theorem provides sufficient conditions for asymptotic stability of the p-th moment of (4.1.1) in the context of the method of variation of parameters.

Theorem 4.4.3. *Assume that the hypotheses of* Theorem 4.4.2 *hold except* (iv) *and* (v) *are replaced by*

(vi) $\sum_{i=1}^{m} |L_s V_i(s, x(t, s, y))| \leq \lambda(s)\eta(t-s) \sum_{i=1}^{m} |V_i(s, y)|$ *for* $t_0 \leq s \leq t$, $\|y\|^p \leq \rho$;

(vii) $$E\left[\sum_{i=1}^{m} |V_i(t_0, x(t, \omega))|\right] \leq \alpha([\|y_0(\omega)\|^p])\beta(t - t_0), \tag{4.4.7}$$

for $E[\|y_0(\omega)\|^p] \leq \rho$, *where* λ *and* α *are as in (iv) and* (v), η, $\beta \in \mathcal{L}$, *and* η *and* β *satisfy condition* (2.4.19). *Then the trivial solution process of* (4.1.1) *is asymptotically stable in the p-th mean.*

Proof. Based on the proofs of Theorems 2.4.3 and 4.4.2, the proof of the theorem can be reformulated. The details are left to the reader.

A corollary similar to Corollary 2.4.1 can be reformulated as as follows.

Corollary 4.4.1. *Let the hypotheses of* Theorem 4.4.3 *be satisfied except that* (2.4.19) *and the condition on* η *are replaced by* (2.4.24) *and*

$$\lim_{t\to\infty} \left[\beta(t - t_0) \exp\left[k \int_{t_0}^{t} \lambda(s) ds\right]\right] = 0. \tag{4.4.8}$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Then the trivial solution process of (4.1.1) is p-th mean asymptotically stable.

Remark 4.4.2. One can formulate remarks similar to Remarks 2.4.5, 2.4.6, and 2.4.7.

Finally, some examples are worked out to show the working mechanism of the method.

Example 4.4.3. Let us consider Example 4.4.2. As in Example 4.1.2, by taking $V(t,x) = \|x\|^2$, we obtain

$$L_s V(s, x(t,s,y)) = tr(\Phi(t,s)B(s)yy^T B^T(s)\Phi^T(t,s)). \tag{4.4.9}$$

Here $x(t,s,y) = \Phi(t,s)y$ is the solution process of (2.1.20) through (s,y). We assume that

$$\|x(t,\omega)\|^2 \leq \|y_0(\omega)\|^2 \exp[-2\gamma(t-t_0)], \tag{4.4.10}$$

where $\gamma > 0$ and $x(t,\omega)$ is the solution process of the average system (2.1.20) of (4.1.18). From (4.4.10), $L_s V(s, x(t,s,y))$ satisfies the following relation

$$L_s V(s, x(t,s,y)) \leq \lambda(s) \exp[-2\gamma(t-s)] V(s,y) \tag{4.4.11}$$

for all $y \in R^n$ and $t_0 \leq s \leq t$, where $\lambda \in C[R_+, R_+]$. We assume the λ in (4.4.11) and γ in (4.4.10) satisfy the following condition

$$\limsup_{t\to\infty} \left[\frac{1}{t-t_0}\int_{t_0}^{t} \lambda(s)ds\right] < 2\gamma. \tag{4.4.12}$$

From the definition of V, $p = 2$, (4.4.9), (4.4.10), (4.4.11) and (4.4.12), all the hypotheses of Corollary 4.4.1 are satisfied. For further clarification, see Example 2.4.3. Now by applying Corollary 4.4.1 and using Example 4.4.2, we conclude that the trivial solution of (4.1.18) is asymptotically stable in the second moment.

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

Example 4.4.4. Let us consider the Itô-type stochastic differential equation

$$dy = -\frac{1}{4}y^5 dt + \sigma(t,y)dz, \qquad y(t_0,\omega) = y_0(\omega) \tag{4.4.13}$$

where $y \in R$ and $z(t)$ is a one-dimensional normalized Wiener process. Assume that $\sigma \in C[R_+ \times R, R]$ and $\sigma(t,0) \equiv 0$ for $t \in R_+$. The mean equation corresponding to (4.4.13) is

$$x' = -\frac{x^5}{4}, \qquad x(t_0,\omega) = y_0(\omega). \tag{4.4.14}$$

It is obvious that $x(t,t_0,y_0(\omega)) = x(t,\omega) = |y_0(\omega)|/[1+(t-t_0)y_0^4(\omega)]^{1/4}$, $\Phi(t,t_0,y_0) = 1/[1+(t-t_0)y_0^4]^{5/4}$, $\frac{\partial^2 x}{\partial x_0^2}(t,t_0,y_0) = -5(t-t_0)|y_0|^3/[1+(t-t_0)y_0^4]^{9/4}$. By taking $V(t,x) = |x|^2$, $L_sV(s,x(t,s,y))$ is given by

$$L_sV(s,x(t,s,y)) = \left(-\frac{5}{2}(t-s)y^4 + 1\right)\sigma^2(s,y)/[1+(t-s)y^4]^{5/2}$$

and assume that it satisfies

$$L_sV(s,x(t,s,y)) \le \lambda(s)y^2,$$

where $\lambda \in C[R_+,R_+] \cap L^1[R_+,R_+]$. As in Example 2.4.4, we note that

$$\begin{aligned} E[|x(t,\omega)|^2] &\le (E[|y_0(\omega)|^2])^{1/2}(E[|y_0|^2/[1+(t-t_0)y_0^4]^{1/2}])^{1/2} \\ &\le (E[|y_0(\omega)|^2])^{1/2}(E[|y_0|^4/[1+(t-t_0)y_0^4])^{1/4} \\ &\le (E[|y_0(\omega)|^2])^{1/2}(E[|y_0(\omega)|^4]/[1+(t-t_0)E[|y^4|]])^{1/4} \\ &\le (E[|y_0(\omega)|^2])^{1/2}(\rho/[1+(t-t_0)\rho])^{1/4} \end{aligned}$$

whenever $E[|y_0(\omega)|^4] \le \rho$ for some $\rho > 0$. From the above discussion, it is clear that (4.4.13) satisfies all the hypotheses of Theorem 4.4.2. Thus

$$E[|y(t,\omega)|^2] \le (E[|y_0(\omega)|^2])^{1/2} + \int_{t_0}^{t} \lambda(s)E[|y(s,\omega)|^2]ds$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

which implies that

$$E[|y(t,\omega)|^2] \leq (E[|y_0(\omega)|^2])^{1/2} \exp\left[\int_{t_0}^{t} \lambda(s)ds\right].$$

From this one can conclude that the trivial solution of (4.4.13) is stable in the mean square sense.

Remark 4.4.3. Examples 2.4.4 and 4.4.4 illustrate that the full force of the unperturbed systems is not used. However, examples (Examples 2.4.1 and 4.4.1) in the context of comparison method have used this force. Again, emphasizing the advantage of comparison method over method of variation of constant parameters.

4.5. ERROR ESTIMATES

We derive the results concerning the error estimates of solutions process of Itô-type stochastic differential systems (4.4.1) with respect to solution process of the corresponding mean system (4.1.1) or (2.1.2).

We state error estimate results in the context of the method of variations for Itô-type system of differential equations (4.1.1).

Theorem 4.5.1. *Let the assumption of* Theorem 4.1.2 *be satisfied. Further assume that*

(i) $b(\|x\|^p) \leq \sum_{i=1}^{m} |V_i(t,x)| \leq a(\|x\|^p)$,

(ii) $\sum_{i=1}^{m} |L_s V_i(s, x(t,s,y) - x(t,s,z))| \leq \lambda_1(s) C(\|y-z\|^p) + \lambda_2(s, \|z\|)$ *and*

(iii) $E[V_x(s, x(t,s,y(s,\omega)) - x(t,s,\bar{x}(s)))\Phi(t,s,y(s,\omega))R(s,y(s,\omega),\omega)]$ *and the continuous time derivative of* $a(E[\|x(t,\omega) - \bar{x}(t)\|^p])$ *exist for* $t \geq t_0$, *where* $a \in \mathcal{CK}$ *and its derivative belongs to* $C[R_+, R_+]$,

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

$b \in \mathcal{VK}$, $C \in \mathcal{CK}$, and $L_s V_i(s, x(t,s,y) - x(t,s,z))$ is the i-th component of

$$L_s V(s, x(t,s,y) - x(t,s,z)) = V_s(s, x(t,s,y) - x(t,s,z))$$
$$+V_x(s, x(t,s,y) - x(t,s,z)) tr\left(\frac{1}{2}\frac{\partial^2 x}{\partial x_0 \partial x_0}(t,s,y) b(s,y)\right)$$
$$+\frac{1}{2} tr(V_{xx}(s, x(t,s,y) - x(t,s,z)) c(t,s,y)), \quad (4.5.1)$$

$p \geq 1$, $\lambda_1 \in C[R_+, R_+]$ and $\lambda_2 \in C[R_+ \times R_+, R_+]$. Let us define $H(s)$, $dH/ds = 1/h(s)$, $h(s) = C(b^{-1}(s))$ and assume that $H \in \mathcal{K}$. Then

$$E[\|y(t,\omega) - \bar{x}(t)\|^p] \leq H^{-1}\left(\int_{t_0}^{t} \lambda_1(s) ds\right.$$
$$\left. +H\left(a(E[\|y_0(\omega) - z_0\|^p]) + \int_{t_0}^{t} \beta(s) ds\right)\right) \quad (4.5.2)$$

for $t \geq t_0$, where $y(t,\omega)$ and $x(t,\omega)$ are the solution processes of (4.1.1) and (2.1.3) through $(t_0, y_0(\omega))$, respectively, and $\bar{x}(t) = x(t, t_0, z_0)$ is the solution process of either (2.1.2) or (2.1.3) depending on the choice of z_0; $\beta(s)$ is the absolute value of the sum of $E[\lambda_2(s, \|\bar{x}(s)\|)]$ and the time derivative of $a(E[\|x(t,\omega) - \bar{x}(t)\|^p])$.

Proof. Let $y(t,\omega)$, $x(t,\omega)$ and $\bar{x}(t)$ be solution processes as described in the theorem. Using Theorem 4.1.2, hypothesis (iii) and taking expected value, we have

$$\sum_{i=1}^{m} E[V_i(t, y(t,\omega) - \bar{x}(t))] = \sum_{i=1}^{m} E[V_i(t_0, x(t,\omega) - \bar{x}(t))]$$
$$+E\left[\int_{t_0}^{t} \sum_{i=1}^{m} L_s V_i(s, x(t,s,y(s,\omega)) - x(t,s,\bar{x}(s)))\right] ds.$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

This together with hypotheses (i), (ii) and the nature of functions, one obtains

$$b(E[\|y(t,\omega)-\bar{x}(t)\|^p]) \leq a(E[\|x(t,\omega)-\bar{x}(t)\|^p]) \\ + \int_{t_0}^{t} (\lambda_1(s)C(E[\|y(s,\omega)-\bar{x}(s)\|^p]) + E[\lambda_2(s,\|\bar{x}(s)\|)])\,ds. \quad (4.5.3)$$

Set

$$R(t) = \int_{t_0}^{t} (C(E[\|y(s,\omega)-\bar{x}(s)\|^p]))\lambda_1(s) + E[\lambda_2(s,\|\bar{x}(s)\|)])ds.$$

Therefore

$$R'(t) = C(E[\|y(t,\omega)-\bar{x}(t)\|^p])\lambda_1(t) + E[\lambda_2(t,\|\bar{x}(t)\|)] \quad R(t_0)=0. \quad (4.5.4)$$

From (4.5.3) we set

$$b(E[\|y(t,\omega)-\bar{x}(t)\|^p]) \leq a(E[\|x(t,\omega)-\bar{x}(t)\|^p]) + R(t), \qquad t \geq t_0.$$

This together with (4.5.4) after certain computation, yields

$$R'(t) \leq h(a(E[\|x(t,\omega)-\bar{x}(t)\|^p]) \\ + R(t))\lambda_1(t) + E[\lambda_2(t,\|\bar{x}(t)\|)], \qquad t \geq t_0$$

where h is as defined in the theorem. By setting

$$u(t) = a(E[\|x(t,\omega)-\bar{x}(t)\|^p]) + R(t)$$

with

$$u(t_0) = a(E[\|y_0(\omega)-z_0\|^p]),$$

$$\beta(t) = |E[\lambda_2(t,\|\bar{x}(t)\|)] + \frac{d}{dt}(a(E[\|x(t,\omega)-\bar{x}(t)\|^p]))|,$$

and imitating the proof of Theorem 2.5.2, the conclusion of theorem follows, immediately. This completes the proof.

A sharper estimate and the feasibility of assumption (ii) with an improved modification is the content of the following result.

Corollary 4.5.1. *Suppose that all the hypotheses of* Theorem 4.5.1 *are satisfied except the differentiability of* $a([\|x(t,\omega) - \bar{x}(t)\|^p])$ *and hypothesis (ii) are replaced by* (2.5.28) *and hypothesis (ii) is valid whenever* (2.5.28) *holds. Then* (4.5.2) *becomes*

$$E[\|y(t,\omega) - \bar{x}(t)\|^p] \le H^{-1}\Bigg(\int_{t_0}^{t} \lambda_1(s)ds + H\left(a(E[\|y_0(\omega) - z_0\|^p]) + \int_{t_0}^{t} E[\lambda_2(s, \|\bar{x}(s)\|)]ds\right)\Bigg). \tag{4.5.5}$$

Proof. The proof of the corollary can be worked out by using the arguments of proofs of Theorem 2.5.2 and Corollary 2.5.1. Details are left to the reader.

The next theorem deals with the modification of hypothesis (ii) of Theorem 4.5.1.

Theorem 4.5.2. *Assume that all the assumptions of* Theorem 4.5.1 *hold except hypothesis* (ii) *is replaced by*

$$\sum_{i=1}^{m} |L_s V_i(s, x(t,s,y) - x(t,s,z))| \le \lambda_1(s) C\left(\sum_{i=1}^{m} |V_i(s, y - z)|\right) + \lambda_2(s, \|z\|), \tag{4.5.6}$$

where $L_s V_i(s, x(t,s,y) - x(t,s,z))$, C, *and* λ_i *are as defined before, and* $h(s) = C(s)$. *Then estimate* (4.5.2) *remains valid.*

Proof. The proof of the theorem follows from the proofs of Theorems 2.5.3 and 4.5.1. Hence, we omit the details.

Corollary 4.5.2. *Let the hypotheses of* Theorem 4.5.2 *be satisfied except the differentiability of* $a(E[\|x(t,\omega) - \bar{x}(t)\|^p])$ *is replaced by* (2.5.28), *and relation* (4.5.6) *is valid whenever* (2.5.28) *holds. Then estimate* (4.5.5) *is true.*

Remark 4.5.1. Remarks corresponding to Remarks 2.5.2, 2.5.3, and 2.5.4 can be formulated, analogously.

Example 4.5.1. Let us consider Example 4.1.2. Here $V(t,x) = \|x\|^2$, $a(s) = b(s) = s$, $p = 2$. From the standard existence theorem relative to system (4.1.18), hypothesis (iii) of Theorem 4.5.1 is valid. It remains to verify hypotheses (ii). In this case

$$L_s V(s, x(t,s,y) - x(t,s,z)) = tr(\Phi(t,s)B(s)yy^T B^T \Phi^T), \quad (4.5.7)$$

where $\Phi(t, t_0)$ is the fundamental matrix solution of the mean differential equation corresponding to (4.1.18). Using properties of the trace of matrices, (4.5.7) can be reduced to

$$\begin{aligned} L_s V(s, x(t,s,y) - x(t,s,z)) &= tr(\Phi(t,s)B(s)yy^T B^T(s)\Phi^T(t,s)) \\ &\le \lambda_1(s)\|y - z\|^2 + \lambda_1(s)\|z\|^2 \end{aligned}$$

provided $\|\Phi(t,s)\| \le K$. Here $\lambda_1 \in C[R_+, R_+]$, $\lambda_2(s, \|z\|) = \lambda_1(s)\|z\|^2$, and $C(s) = s$. This verifies all the hypotheses of Corollary 4.5.2, and hence (4.5.5) becomes

$$\begin{aligned} &E[\|y(t,\omega) - \bar{x}(t)\|^2] \le \\ &\qquad (E[\|y_0(\omega) - z_0\|^2] + \int_{t_0}^t E[\lambda_2(s, \|\bar{x}(s)\|)]ds) \exp\left[\int_{t_0}^t \lambda_1(s)ds\right]. \end{aligned}$$

Furthermore, from (2.5.28) and definition of λ_2, the above inequality reduces to

$$\begin{aligned} &E[\|y(t,\omega) - \bar{x}(t)\|^2] \le \\ &\qquad \left(E[\|y_0(\omega) - z_0\|^2] + K^2 E[\|z_0\|^2] \int_{t_0}^t \lambda_1(s)ds\right) \exp\left[\int_{t_0}^t \lambda_1(s)ds\right]. \end{aligned}$$

Remark 4.5.2. In many situations additional information is lost if we try to verify (2.5.28). As a result of this our error estimates lead to greater value than one may expect. For example, if $\Phi(t, t_0)$ is the fundamental solution of (2.1.20) which is the mean of (4.1.18). If we assume that

$$\|\Phi(t, t_0)\| \leq K \exp[-\alpha(t - t_0)], \qquad \text{for } t \geq t_0 \tag{4.5.8}$$

where $\alpha > 0$. In this case (2.5.28) is satisfied but an additional information about the exponential decaying nature is lost in our consideration. To preserve this information, certain careful computations are needed to achieve greater degree of accuracy in our error estimates. This will be shown in the following example.

Example 4.5.2. Let us consider again Example 4.1.2. We assume that $\Phi(t, s)$ satisfies (4.5.8). For the choice of $V(t, x) = \|x\|^2$, by following the discussion in Example 4.5.1 in the context of (4.5.8) we compute

$$L_s V(s, x(t, s, y) - x(t, s, z)) \leq \\ K^2 \exp[-2\alpha t]\Big(e^{2\alpha s}\|y - z\|^2 + e^{2\alpha s}\|z\|^2\Big)\lambda(s).$$

This together with the relation (4.1.14) in Example 4.1.2, we get

$$E[V(t, y(t, \omega) - \bar{x}(t))] \leq E[V(t_0, x(t, \omega) - \bar{x}(t))] \\ + K^2 \exp[-2\alpha t] \int_{t_0}^{t} e^{2\alpha s} E\Big(V(s, y(s, \omega) - \bar{x}(s)) \\ + V(s, \bar{x}(s))\Big)\lambda(s) ds$$

for $t \geq t_0$. This implies

$$\exp[2\alpha t] E[V(t, y(t, \omega) - \bar{x}(t))] \leq \exp[2\alpha t] E[V(t_0, x(t, \omega) - \bar{x}(s))] \\ + K^2 \int_{t_0}^{t} \lambda(s)(e^{2\alpha s} E[V(s, y(s, \omega) - \bar{x}(s))] + e^{2\alpha s} E[V(s, \bar{x}(s))]) ds. \tag{4.5.9}$$

By setting

$$R(t) = K^2 \int_{t_0}^{t} (e^{2\alpha s}E[V(s, y(s,\omega) - \bar{x}(s))] + e^{2\alpha s}E[V(s, \bar{x}(s))])\lambda(s)ds$$

and differentiating,

$$R'(t) = K^2(e^{2\alpha t}E[V(t, y(t,\omega) - \bar{x}(t))] + e^{2\alpha t}E[V(t, \bar{x}(t))])\lambda(t),$$

$$R(t_0) = 0.$$

From (4.5.9) we get

$$e^{2\alpha t}E[V(t, y(t,\omega) - \bar{x}(t))] \leq e^{2\alpha t}E[V(t_0, \|x(t,\omega) - \bar{x}(t)\|)] + R(t), \tag{4.5.10}$$

$t \geq t_0$ and hence

$$R'(t) \leq \lambda(t)K^2(e^{2\alpha t}E[V(t_0, x(t,\omega) - \bar{x}(t))]$$
$$+R(t)) + K^2 e^{2\alpha t}E[V(t, \bar{x}(t))]\lambda(t).$$

This together with (4.5.8), we have

$$R'(t) \leq \lambda(t)R(t) + \lambda(t)K^4(E[\|y_0(\omega) - z_0\|^2 + E[\|z_0\|^2])$$

with $R(t_0) = 0$. From this and scalar version Theorem A.2.4, we have

$$R(t) \leq K^4(E[\|y_0(\omega) - z_0\|^2] + E[\|z_0\|^2]) \int_{t_0}^{t} \lambda(s) \exp\left[\int_{s}^{t} \lambda(u)du\right] ds.$$

This together with (4.5.10) yields

$$E[V(t, y(t,\omega) - \bar{x}(t))] \leq E[V(t_0, x(t,\omega) - \bar{x}(t))]$$
$$+ \exp[-2\alpha t]K^4(E[\|y_0(\omega) - z_0\|^2 + \|z_0\|^2]) \int_{t_0}^{t} \lambda(s) \exp\left[\int_{s}^{t} \lambda(u)du\right] ds$$

and hence

$$E[\|y(t,\omega) - \bar{x}(t)\|^2] \leq K^2 \exp(-2\alpha t)\Bigg[e^{2\alpha t_0}E[\|y_0(\omega) - z_0\|^2]$$
$$+ K^2(E[\|y_0(\omega) - z_0\|^2] + \|z_0\|^2) \int_{t_0}^{t} \lambda(s) \exp\left[\int_{s}^{t} \lambda(u)du\right] ds\Bigg]. \tag{4.5.11}$$

This estimate is then the final estimate of Example 4.5.2.

In the following, we present an error estimate result in the context of comparison theorems.

Theorem 4.5.3. *Let the hypotheses of* Theorem 4.2.2 *be satisfied. Further assume that*

$$b(\|x\|^p) \le \sum_{i=1}^{m} V_i(t,x), \tag{4.5.12}$$

where $b \in \mathcal{CK}$ *and it is convex function and* $p \ge 1$. *Then*

$$b(E[\|y(t,\omega) - \bar{x}(t)\|]^p) \le \sum_{i=1}^{m} E[r_i(t,t_0,V(t_0,x(t,\omega)))], \qquad t \ge t_0. \tag{4.5.13}$$

Proof. The proof of the theorem can be formulated based on the proofs of Theorems 4.2.2 and 2.5.1. The details are left to the reader.

4.6. RELATIVE STABILITY

The sufficient conditions for relative stability of Itô type system with its mean are given in the following result.

Theorem 4.6.1. *Let the hypotheses of* Theorem 4.2.2 *be satisfied. Further, suppose that for* $(t,x) \in R_+ \times R^n$

$$b(\|x\|^p) \le \sum_{i=1}^{m} V_i(t,x) \le a(t, \|x\|^p),$$

where $b \in \mathcal{VK}$, $a \in \mathcal{CK}$, *and* $p \ge 1$. *Then*

(i) (JR_1) *of* (4.2.3) *and* (2.1.2) *implies* (RM_1) *of* (4.1.1) *and* (2.1.2);

(ii) (JR_2) *of* (4.2.3) *and* (2.1.2) *implies* (RM_2) *of* (2.1.23) *and* (2.1.2).

Proof. The proof of the theorem can be worked out by imitating the proof of Theorem 2.6.1 in the context of Theorem 4.2.2 and Remark 2.2.1.

The following example justifies the usefulness of Theorem 4.6.1 and joint relative stability concept.

Example 4.6.1. Consider system (4.1.18) and its mean systems (2.1.19) and (2.1.20). Let $V(t,x) = \frac{1}{2}\|x\|^2$. In this case, $a(r) = b(r) = r$, and the comparison equation is as given by (4.2.17). To apply Theorem 4.6.1, we need to verify the joint relative stability in the mean of (4.2.17) and (2.1.19) (or (2.1.20)). For this purpose, we assume that $\mu(\widehat{A}(t)) + \lambda(t)$ is integrable on $[t_0, +\infty)$. In this case (4.2.17) and (2.1.19) satisfy (JR_1) property. Hence, by the application of Theorem 4.6.1, systems (4.1.18) and (2.1.19) are mean square relatively stable. On the other hand, if we assume that $\mu(\widehat{A}(t)) + \lambda(t)$ satisfy relation (2.6.8), then systems (4.2.17) and (2.1.19) are jointly relatively asymptotically stable in the mean. Again by Theorem 4.6.1, systems (4.1.18) and (2.1.19) are relatively asymptotically stable in the mean square.

The sufficient conditions for relative stability of Itô type system of differential equations in the context of method of variation of parameters are given in the next result.

Theorem 4.6.2. *Let the assumption of* Theorem 4.1.2 *be satisfied. Further assume that*

(i) $\widehat{F}(t,0) \equiv 0$ *for* $t \in R_+$;

(ii) $b(\|x\|^p) \le \sum_{i=1}^m |V_i(t,x)| \le a(\|x\|^p)$ *for* $(t,x) \in R_+ \times R^n$, $b \in \mathcal{VK}$, $a \in \mathcal{CK}$, $p \ge 1$;

(iii) $\sum_{i=1}^m |L_s V_i(s, x(t,s,y) - x(t,s,z))| \le$ $\eta(t-s)\left[\lambda_1(s)\left(\sum_{i=1}^m |Vi(s,y-z)|\right) + \lambda_2(s)\left(\sum_{i=1}^m |V_i(s,z)|\right)\right]$ *for* $t_0 \le s \le t$, $\|y-z\|^p < \rho$, $\|z\|^p < \rho$ *some* $\rho > 0$, λ_i *are locally integrable functions and* $\eta \in \mathcal{L}$;

(iv) $\sum_{i=1}^m |V_i(t_0, y(t,\omega) - \bar{x}(t))| \le \alpha(\|y_0(\omega) - z_0\|^p)\beta(t-t_0)$, $t \ge t_0$, *whenever* $E[\|y_0 - z_0\|^p] < \rho$ *and* $E[\|z_0\|^p] < \rho$, *where* $\alpha \in \mathcal{CK}$,

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

$\beta \in \mathcal{L}$;

(v) *there exists a positive number* K *such that*

$$\eta(t-s)\beta(s-t_0) \le K\beta(t-t_0)$$

where η *and* β *are as defined above; Then*

(1) *the boundedness of* $\beta(t-t_0)\exp\left[K\int_{t_0}^t \lambda_1(s)ds\right]$ *and*

$$\beta(t-t_0)\int_{t_0}^t \lambda_2(s)\exp\left[K\int_s^t \lambda_1(\nu)d\nu\right]ds$$

imply (RM_1) *of* (4.1.1) *and* (2.1.2);

(2) $\lim_{t\to\infty}\left[\beta(t-t_0)\exp\left[K\int_{t_0}^t \lambda_1(s)ds\right]\right] = 0$ *and*

$$\lim_{t\to\infty}\left[\beta(t-t_0)\int_{t_0}^t \lambda_2(s,\omega)\exp\left[\int_s^t \lambda_1(s)ds\right]\right] = 0$$

imply (RM_2) *of* (4.1.1) *and* (2.1.2).

Proof. The proof of the theorem can be reconstructed by following the arguments of Theorems 4.4.2 and 2.6.2. The details are left as an exercise.

Finally, we present an example to illustrate the importance of Theorem 4.6.2.

Example 4.6.2. Let us consider Example 4.5.2. From the discussion of this example, it is obvious that hypotheses (i)–(v) of Theorem 4.6.2 are satisfied with $V(t,x) = \|x\|^2$, $\eta(t-s) = \beta(t-s) = K^2\exp[-2\alpha(t-s)]$, $\lambda_1 = \lambda_2 = \lambda$, $p = 2$, and a, b and α in the theorem are identity functions. If we assume that

$$\exp\left[-2\alpha(t-t_0) + \int_{t_0}^t \lambda(s)ds\right]$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

is bounded and

$$\lim_{t\to\infty}\left[\frac{1}{t-t_0}\int_{t_0}^{t}\lambda(s)ds\right] < 2\alpha,$$

then (4.1.18) and (2.1.19) have (RM_1) and (RM_2) properties, respectively.

Remark 4.6.1. A remark similar to Remark 2.6.1 can be formulated.

4.7. APPLICATIONS TO POPULATION DYNAMICS

In this section, we consider the mathematical model of population growth of n-species community under random environmental fluctuations. A mathematical model of n-species community similar to the mathematical model described by (2.7.1) will be considered. In the present discussion, the random parameters $a_i(\omega)$ and $b_{ij}(\omega)$ in (2.7.1) for $i,j = 1,2,\ldots,n$ are replaced by time varying stochastic processes $a_i = \bar{a}_i + \eta_i(t)$ and $b_{ij} = \bar{a}_i + \eta_{ij}(t)$ for $i,j = 1,2,\ldots,n$. Here for $i,j = 1,2,\ldots,n$, $\eta_i(t)$ are $\eta_{ij}(t)$ are white noise processes and are completely correlated for each value of i. Furthermore, they are further described by

$$\begin{cases} \eta_i(t)dt = \sigma_i(N)\bar{a}_i dz(t) \\ \eta_{ij}(t)dt = \sigma_i(N)\bar{b}_{ij} dz(t) \end{cases} \tag{4.7.1}$$

where $i,j = 1,2,\ldots,n$, $z(t)$ is a normalized Wiener process, $\bar{a}_i$ and $\bar{b}_{ij}$ are deterministic real numbers with $\bar{a}_i$ and $\bar{b}_{ii}$ being positive; $\sigma_i \in C[R^n, R]$. The processes $\eta_i(t)$ and $\eta_{ij}(t)$ characterize the immigration/emigration process in the population dynamics of n-species community. From the above discussion the mathematical model described in (2.7.1) can be rewritten as

$$N_i' = N_i\left(\bar{a}_i + \eta_i(t) - \sum_{j=1}^{n}\left(\bar{b}_{ij} + \eta_{ij}(t)\right)N_j\right), \qquad N_i(t_0,\omega) = N_{i0}(\omega) \tag{4.7.2}$$

for $i = 1, 2, \ldots, n$. From (4.7.1), (4.7.2) can be further represented by the Itô-type system of equations,

$$dN_i = N_i \left(\bar{a} - \sum_{j=1}^{n} \bar{b}_{ij} N_j \right) dt$$
$$+\sigma_i(N) N_i \left(\bar{a}_i - \sum_{j=1}^{n} \bar{b}_{ij} N_j \right) dz(t), \quad N_i(t_0) = N_{i0}(\omega). \quad (4.7.3)$$

Suppose that the non-trivial equilibrium population densities of the species in the community described by (4.7.3) are $\bar{\alpha}_i$, $i = 1, 2, \ldots, n$. Again using the transformation $y_i = N_i - \bar{\alpha}_i$, $i = 1, 2, \ldots, n$, system (4.7.3) can be written as

$$dy = (\bar{B}(\bar{\alpha})y - Y(y)\bar{B}y)dt$$
$$+\bar{\sigma}(y)(\bar{B}(\bar{\alpha})y - Y(y)\bar{B}y)dz, \quad y(t_0, \omega) = y_0(\omega), \quad (4.7.4)$$

where $y_{i0}(\omega) = N_{i0}(\omega) - \bar{\alpha}_i$; $\bar{B}(\bar{\alpha})$ and $\bar{B}$ are as described in (2.7.9), $Y(y) = \mathrm{diag}(y_1, y_2, \ldots, y_n)$ and $\bar{\sigma}(y) = \mathrm{diag}(\sigma_1(y + \bar{\alpha}), \sigma_2(y + \bar{\alpha}), \ldots, \sigma_n(y + \bar{\alpha}))$ are matrices; $z(t)$ are 1-dimensional normalized Wiener process. We further assume that σ_i's for $i = 1, 2, \ldots, n$ satisfy condition

$$|\bar{\sigma}_i(y) y_i y_j| \leq \sigma^* \quad (4.7.5)$$

for some positive constant σ^*. Furthermore, they are smooth enough to insure the existence and uniqueness of solution process of (4.7.4).

The smooth systems corresponding to initial value problem in (4.7.4) both deterministic and stochastic initial conditions are as described in (2.7.9) and (2.7.10), respectively.

Let $y(t, \omega)$ be the solution process of (4.7.4) and let $\bar{x}(t) = x(t, t_0, z_0)$ be the solution process of either (2.7.9) of (2.7.10) depending on the initial data (t_0, m_0) or $(t_0, x_0(\omega))$. Furthermore, for

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

$x_0(\omega) = y_0(\omega)$, $\bar{x}(t)$ is represented by $x(t,\omega) = x(t, t_0, y_0(\omega))$. With this formulation, we are ready to apply the variation of constants method of Section 4.1.1 to investigate the stability of the equilibrium solution process of (4.7.4). The following result provides the sufficient conditions for the mean-square stability of the trivial solution process of (4.7.4).

Theorem 4.7.1. *Assume that $\bar{\sigma}(y)$ satisfies condition* (4.7.5) *and it is smooth enough to insure the existence and uniqueness of the solution process of* (4.7.4). *Further, assume that*

(i) $\frac{\partial x}{\partial x_0}(t, s, y(s,\omega)) = \Phi(t, s, y(s,\omega))$ *of* (2.7.10) *satisfies*

$$\|\Phi^T(t, s, y(s,\omega))\Phi(t, s, y(s,\omega))\| \le \exp[-2\beta(t-s)]; \qquad t_0 \le s \le t,$$

and β is a positive real number;

(ii) $E[\|y_0(\omega)\|^2] \le \rho$, *for some $\rho > 0$;*

(iii) $\beta > K$ *where β is as defined in* (i) *and $K = \sum_{i,j=1}^{n}(\gamma_{ij} + \mu_{ij})c_{ij}$, here γ_{ij} and μ_{ij} are given by the following inequalities,*

$$E\left[\left(x^T(t, s, y(s,\omega))\,\frac{\partial^2 x}{\partial x_{0i} x_{0j}}\,(t, s, y(s,\omega))\delta_{ij}(s, y(s,\omega))\right)\right] \le \gamma_{ij} c_{ij} \exp[-2\beta(t-s)]E[\|y(s,\omega)\|^2];$$

and

$$E\left[\sum_{k=1}^{n}\left(\frac{\partial x_k}{\partial x_{0i}}\,(t, s, y(s,\omega))\frac{\partial x_k}{\partial x_{0j}}\,(t, s, y(s,\omega))\right)\delta_{ij}(s, y(s,\omega))\right] \le \mu_{ij} c_{ij} \exp[-2\beta(t-s)]E[\|y(s,\omega)\|^2]$$

where

$$(\delta_{ij}(s, y(s,\omega))) = [\bar{\sigma}(y(s,\omega))\widehat{F}(s, y(s,\omega))][\bar{\sigma}(y(s,\omega))\widehat{F}(s, y(s,\omega))]^T.$$

Then, the solution process (4.7.4) *is asymptotically stable in the mean square sense.*

Proof. Consider the solution process $x(t, s, y(s, \omega))$ of (2.7.10) and $y(t, \omega) = y(t, t_0, y_0(\omega))$ of (4.7.4) through $(s, y(s, \omega))$ and $(t_0, y_0(\omega))$, respectively. The function $\widehat{F}(t, x)$ is continuously differentiable with respect to t and twice continuously differentiable with respect to x_0. From this discussion and applying Theorem 4.1.1 for $V(t, x) = \|x\|^2$, one can obtain the total derivative of $\|x(t, s, y(s, \omega))\|^2$:

$$
\begin{aligned}
d_s\|x(t, s, y(s, \omega))\|^2 = {} & \\
& 2x^T(t, s, y(s, \omega))\Phi(t, s, y(s, \omega))\bar{\sigma}(y, (s, \omega))\widehat{F}(s, y(s, \omega))dz(s) \\
& + \sum_{i,j=1}^{n}\Bigg[\Bigg(x^T(t, s, y(s, \omega))\frac{\partial^2 x}{\partial x_{0i}x_{0j}}(t, s, y(s, \omega)) \\
+ \sum_{k=1}^{n}\left(\frac{\partial x_k}{\partial x_{0i}}(t, s, y(s, \omega))\frac{\partial x_k}{\partial x_{0j}}(t, s, y(s, \omega))\right)\Bigg)\delta_{ij}(s, y(s, \omega))\Bigg], \quad (4.7.6)
\end{aligned}
$$

where $\delta_{ij}(s, y(s, \omega)) = [\bar{\sigma}(y(s, \omega))\widehat{F}(s, y(s, \omega))][\bar{\sigma}(y(s, \omega))\widehat{F}(s, y(s, \omega))]^T$.

Integrating from t_0 to t, we obtain

$$
\begin{aligned}
\|y(t, \omega)\|^2 = {} & \\
& \|x(t, \omega)\|^2 + 2\int_{t_0}^{t} x^T(t, s, y(s, \omega))\Phi(t, s, y(s, \omega))\sigma(s, y(s, \omega))dz(s) \\
& + \sum_{i,j=1}^{n}\Bigg[\int_{t_0}^{t}\Bigg(x^T(t, s, y(s, \omega))\frac{\partial^2 x}{\partial x_{0i}x_{0j}}(t, s, y(s, \omega)) \\
+ \sum_{k=1}^{n}\left(\frac{\partial x_k}{\partial x_{0i}}(t, s, y(s, \omega))\frac{\partial x_k}{\partial x_{0j}}(t, s, y(s, \omega))\right)\Bigg)\delta_{ij}(s, y(s, \omega))ds\Bigg]. \quad (4.7.7)
\end{aligned}
$$

Taking the expectation both sides of (4.7.7),

$$
\begin{aligned}
& E[\|y(t, \omega)\|^2] = E[\|x(t, \omega)\|^2] \\
& +2E\left[\int_{t_0}^{t} x^T(t, s, y(s, \omega))\Phi(t, s, y(s, \omega))\bar{\sigma}(y(s, \omega))\widehat{F}(s, y(s, \omega))dz(s)\right] \\
& \quad + \sum_{i,j=1}^{n} E\Bigg[\int_{t_0}^{t}\Bigg(x^T(t, s, y(s, \omega))\frac{\partial^2 x}{\partial x_{0i}x_{0j}}(t, s, y(s, \omega))
\end{aligned}
$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

$$+\sum_{k=1}^{n}\left(\frac{\partial x_k}{\partial x_{0i}}(t,s,y(s,\omega))\frac{\partial x_k}{\partial x_{0j}}(t,s,y(s,\omega))\right)\Big)\delta_{ij}(s,y(s,\omega))ds\Bigg]. \tag{4.7.8}$$

Using (4.7.5) and (i), the properties of $\frac{\partial^2 x}{\partial x_{0i}x_{0j}}(t,s,y(s,\omega))$ and $\delta_{ij}(s,y(s))$, (4.7.8) reduces to

$$E[\|y(t,\omega)\|^2] \leq E[\|y_0(\omega)\|^2]e^{-2\beta(t-t_0)} + \int_{t_0}^{t}\sum_{i,j=1}^{n}(\gamma_{ij}+\mu_{ij})c_{ij}e^{-2\beta(t-s)}E[\|y(s,\omega)\|^2]ds. \tag{4.7.9}$$

Let $u(t) = E[\|y(t,\omega)\|^2]e^{2\beta(t-t_0)}$. Then (4.7.9) becomes

$$u(t) \leq E[\|y_0(\omega)\|^2] + \int_{t_0}^{t} 2Ku(s)ds. \tag{4.7.10}$$

Using Gronwall's inequality,

$$E[\|y(t,\omega)\|^2] \leq E[\|y_0(\omega)\|^2]e^{-2(\beta-K)(t-t_0)}. \tag{4.7.11}$$

Using (ii), (iii) and (4.7.11), we can conclude that the solution process of (4.7.4) is asymptotically stable in the mean square sense.

Remark 4.7.1. $\bar{\sigma}(x)$ defined in Theorem 4.7.1 characterizes the effects of the randomness. It is a part of a fraction of the variance. The condition $|\sigma_i(x)x_ix_j| \leq \sigma^*$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,n$ reflects that the variance does not grow indefinitely as the density of the population grows.

Remark 4.7.2. The condition (iii), $\beta > K$ characterizes the effects of randomness on the stability of the Lotka-Volterra model of n-species. In particular, it gives the tolerance for the stability of the community in the absence of randomness. In other words, to what extent the stability of the n-species community is preserved under random environmental fluctuations. Further, we note that the randomness in the system is a destabilizing agent in the context of mean square sense.

Example 4.7.1. Consider the following so-called solvable models studied by Montroll [94] of n-interacting species,

$$du_i = u_i^{1-\alpha}\left(\frac{u_i^\alpha - 1}{\alpha}\right)\left[\bar{a}_i - \sum_{j=1}^{n}\bar{b}_{ij}\left(\frac{u_j^\alpha - 1}{\alpha}\right)\right]dt$$
$$+u_i^{1-\alpha}\left[\bar{a}_i - \sum_{j=1}^{n}\bar{b}_{ij}\left(\frac{u_j^\alpha - 1}{\alpha}\right)\right]\sigma_i\left(\frac{u^\alpha - 1}{\alpha}\right)dz(t), \quad (4.7.12)$$

$i = 1,2,3,\ldots,n$ and $u^\alpha - 1 = (u_1^\alpha - 1, u_2^\alpha - 1, \ldots, u_n^\alpha - 1)$. When $\alpha \to 0$, this model reduces to

$$dN_i = N_i\left(\bar{a}_i - \sum_{j=1}^{n}\bar{b}_{ij}N_j\right)dt$$
$$+\left(\bar{a}_i - \sum_{j=1}^{n}\bar{b}_{ij}N_j\right)\sigma_i(N)dz(t); \quad i = 1,2,3,\ldots,n, \quad (4.7.13)$$

where $N_i = \ln u_i$, $N = (N_1, N_2, \ldots, N_n)$. When $\alpha = 1$, this model reduces to

$$dN_i = N_i\left(\bar{a}_i - \sum_{j=1}^{n}\bar{b}_{ij}N_j\right)dt$$
$$+\left(\bar{a}_i - \sum_{j=1}^{n}\bar{b}_{ij}N_j\right)\sigma_i(N)dz(t); \quad i = 1,2,3,\ldots,n, \quad (4.7.14)$$

where $N_i = u_i - 1$, $i = 1,2,3,\ldots,n$. Equation (4.7.12) can be rewritten as

$$u_i^{\alpha-1}du_i = \left(\frac{u_i^\alpha - 1}{\alpha}\right)\left[\bar{a}_i - \sum_{j=1}^{n}\bar{b}_{ij}\left(\frac{u_j^\alpha - 1}{\alpha}\right)\right]dt$$
$$+\left[\bar{a}_i - \sum_{j=1}^{n}\bar{b}_{ij}\left(\frac{u_j^\alpha - 1}{\alpha}\right)\right]\sigma_i\left(\frac{u^\alpha - 1}{\alpha}\right)dz(t), \quad (4.7.15)$$

$$d\left(\frac{u_i^\alpha - 1}{\alpha}\right) = \left(\frac{u_i^\alpha - 1}{\alpha}\right)\left[\bar{a}_i - \sum_{j=1}^{n} \bar{b}_{ij}\left(\frac{u_j^\alpha - 1}{\alpha}\right)\right] dt$$
$$+ \left[\bar{a}_i - \sum_{j=1}^{n} \bar{b}_{ij}\left(\frac{u_j^\alpha - 1}{\alpha}\right)\right] \sigma_i\left(\frac{u^\alpha - 1}{\alpha}\right) dz(t). \quad (4.7.16)$$

Set $N_i = (u_i^\alpha - 1)/\alpha$, Equation (4.7.16) becomes

$$dN_i = N_i\left(\bar{a}_i - \sum_{j=1}^{n} \bar{b}_{ij} N_j\right) dt$$
$$+ \left(\bar{a}_i - \sum_{j=1}^{n} \bar{b}_{ij} N_j\right) \sigma_i(N) dz(t); \qquad i = 1, 2, 3, \ldots, n. \quad (4.7.17)$$

Let N^* be the non-trivial equilibrium points of (4.7.17), and will be given by

$$\bar{a}_i - \sum_{j=1}^{n} \bar{b}_{ij} N_j^* = 0; \qquad i = 1, 2, 3, \ldots, n.$$

Set $y_i = N_i - N_i^*$, then (4.7.17) becomes,

$$dy_i = (y_i + N_i^*)\left(-\sum_{j=1}^{n} \bar{b}_{ij} y_j\right) dt$$
$$+ \left(-\sum_{j=1}^{n} \bar{b}_{ij} y_j\right) \sigma_i(y) dz(t); \quad i = 1, 2, 3, \ldots, n. \quad (4.7.18)$$

In matrix form, (4.7.18) can be written as

$$dy = [\bar{B}(N^*)u - Y(y)\bar{B}y]dt + \Lambda(y)dz, \qquad y_0(\omega) = N(t_0, \omega) - N^*, \quad (4.7.19)$$

where $\bar{B} = (\bar{b}_{ij})_{n\times n}$ and

$$\Lambda(y) = \mathrm{diag}(\sigma_1(y), \sigma_2(y), \ldots, \sigma_n(y))\bar{B}y$$

$$Y(y) = \mathrm{diag}(y_1, y_2, \ldots, y_n).$$

By assuming that $\|y\Lambda^T(y)\Lambda(y)\| \le K(1+\|y\|^2)$, and following the steps of the proof of Theorem 4.7.1, we can conclude that the system (4.7.12) is asymptotically stable in the mean square sense.

4.8. NUMERICAL EXAMPLES

To illustrate the results of Section 4.5, we present some numerical examples. We apply the error estimates found in Examples 4.5.1 and 4.5.2 to the stochastic initial value problem

$$dy = -ydt + \frac{0.1}{1+t}ydz, \qquad y(0,\omega) = y_0(\omega) \tag{4.8.1}$$

and the corresponding deterministic initial value problem

$$m' = -m, \qquad E[y_0(\omega)] = m(0) = m_0, \tag{4.8.2}$$

where $y \in R$ and $z(t)$ is a one-dimensional normalized Wiener process. We assume that $y_0(\omega)$ is Gaussian with variance 0.0001. We use Maple to generate sample solutions of (4.8.1) when $E[y_0(\omega)] = 0.25$, 0.5, and 0.75. From these numerical results we calculate $E[\|y(t,\omega) - m(t)|^2]$ for $t = 0.1, 0.2, \ldots, 1.0$. In this case, for $K = e^{0.01}$ and $\lambda_1(s) = \frac{0.02}{(1+s)^2}$, the error estimate in Example 4.5.1 with respect to (4.8.1) reduces to

$$E[\|y(t,\omega) - m(t)|^2] \le \\ \left(0.0001 + e^{0.02}m_0^2 \int_0^t \frac{0.02}{(1+s)^2}ds\right) \exp\left[\int_0^t \frac{0.02}{(1+s)^2}ds\right]. \tag{4.8.3}$$

Furthermore, for $K = e^{0.01}$, $\alpha = 1$, and $\lambda(s) = \frac{0.01}{(1+s)^2}$, the error estimate in Example 4.5.2 with respect to (4.8.1) becomes

$$E[\|y(t,\omega) - m(t)|^2] \le e^{0.02}e^{-2t}0.0001 \\ +e^{-2t}e^{0.04}(0.0001 + m_0^2) \int_0^t \frac{0.01}{(1+s)^2} \exp\left[\int_s^t \frac{0.01}{(1+u)^2}du\right]ds. \tag{4.8.4}$$

The numerical results shown in Tables 4.8.1, 4.8.2, and 4.8.3 support Remark 4.5.2 that the error estimate obtained in Example 4.5.2 is superior to the one in Example 4.5.1. This is because of assumption (4.5.8).

Table 4.8.1. $E[|y(t,\omega) - m(t)|^2]$ and its bounds (4.8.3) and (4.8.4) with $m_0 = 0.25$

$m_0 = 0.25$			
t	$E[\|y(t,\omega) - m(t)\|^2]$	bound (4.8.3)	bound (4.8.4)
0.1	.000106	.000216	.000132
0.2	.000122	.000314	.000141
0.3	.000112	.000396	.000139
0.4	.000103	.000467	.000130
0.5	.000104	.000529	.000118
0.6	.000094	.000583	.000104
0.7	.000080	.000630	.000091
0.8	.000071	.000673	.000079
0.9	.000060	.000710	.000068
1.0	.000052	.000745	.000058

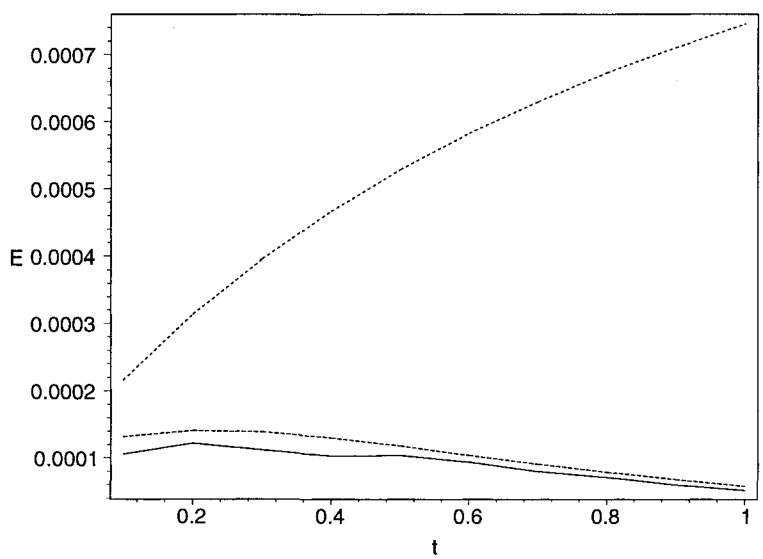


Figure 4.8.1. $m_0 = 0.25$: $E[|y(t,\omega) - m(t)|^2]$ is the solid curve, its bound (4.8.3) is the upper dashed curve, and its bound (4.8.4) is the lower dashed curve.

Table 4.8.2. $E[|y(t,\omega) - m(t)|^2]$ and its bounds (4.8.3) and (4.8.4) with $m_0 = 0.5$

$m_0 = 0.5$			
t	$E[\lvert y(t,\omega) - m(t)\rvert^2]$	bound (4.8.3)	bound (4.8.4)
0.1	.000217	.000565	.000277
0.2	.000314	.000953	.000359
0.3	.000335	.001283	.000386
0.4	.000315	.001566	.000380
0.5	.000305	.001812	.000357
0.6	.000270	.002028	.000325
0.7	.000199	.002219	.000290
0.8	.000190	.002388	.000255
0.9	.000186	.002540	.000221
1.0	.000162	.002677	.000190

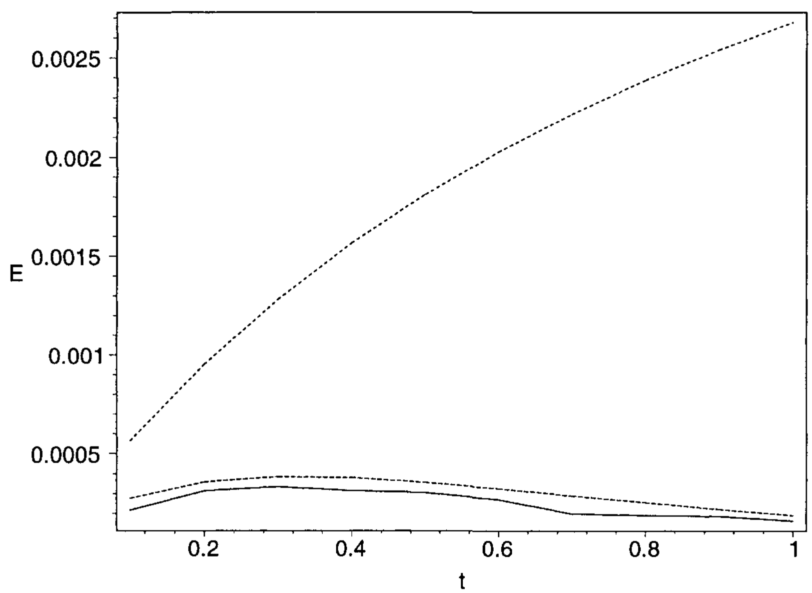


Figure 4.8.2. $m_0 = 0.5$: $E[|y(t,\omega) - m(t)|^2]$ is the solid curve, its bound (4.8.3) is the upper dashed curve, and its bound (4.8.4) is the lower dashed curve.

Table 4.8.3. $E[|y(t, \omega) - m(t)|^2]$ and its bounds (4.8.3) and (4.8.4) with $m_0 = 0.75$

$m_0 = 0.75$			
t	$E[\lvert y(t,\omega) - m(t)\rvert^2]$	bound (4.8.3)	bound (4.8.4)
0.1	.000473	.001145	.000520
0.2	.000561	.002020	.000723
0.3	.000644	.002761	.000798
0.4	.000642	.003399	.000799
0.5	.000646	.003952	.000757
0.6	.000505	.004437	.000693
0.7	.000437	.004866	.000621
0.8	.000374	.005247	.000547
0.9	.000314	.005589	.000476
1.0	.000256	.005897	.000411

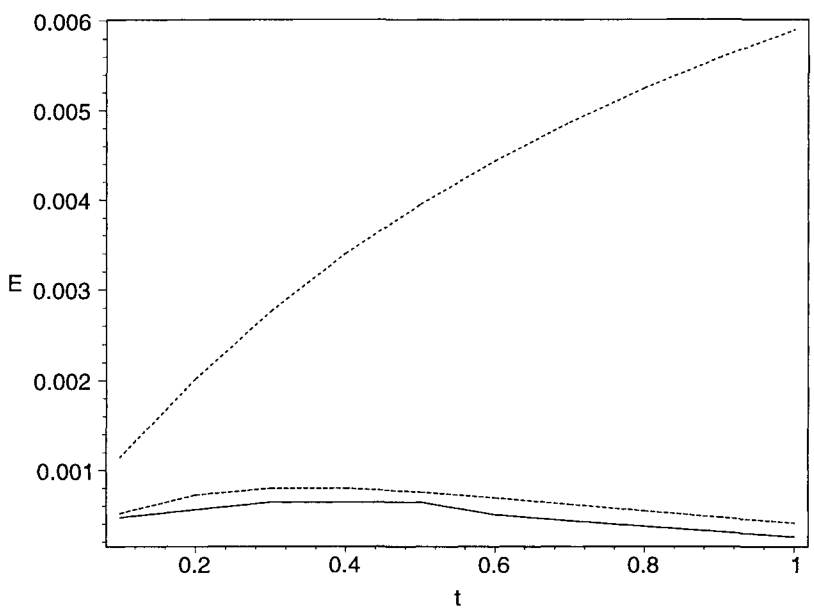


Figure 4.8.3. $m_0 = 0.75$: $E[|y(t, \omega) - m(t)|^2]$ is the solid curve, its bound (4.8.3) is the upper dashed curve, and its bound (4.8.4) is the lower dashed curve.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

4.9. NOTES AND COMMENTS

The contents of Sections 4.1 and 4.2 are due to Ladde [65]. Corollaries 4.1.1 and 4.2.1 are well-known results of Kulkarni and Ladde [55] and Ladde [58], respectively. See Ladde and Lakshmikantham [67]. The results of Section 4.3 are based on the results in Soong [104]. See also Ladde and Robinson [80]. The contents of Sections 4.4 and 4.5 are based on the work of Ladde [65]. The results of Section 4.6 are new. Theorem 4.7.1 is based on the results of Ladde and Sathananthan [76]. Example 4.7.1 is constructed on the basis of the ideas and model of Montroll [94]. See also Feldman and Roughgarden [33], Fleming and Rishel [34], Goel and Richter-Dyn [41], Ladde and Robinson [79], Levins [87], Levontin and Cohen [88], and May [91].

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

CHAPTER 5: BOUNDARY VALUE PROBLEMS OF ITÔ-TYPE

5.0 INTRODUCTION

This chapter deals with a stochastic boundary value problem of Itô-type. It is in the spirit of the main goal that is emphasized in the earlier chapters. This topic is at its infancy level. There are several problems open in this topic.

By employing the Green's function method, the existence of pth mean ($p \geq 1$) solution process of stochastic boundary value problem of Itô-type is established in Section 5.1. The mean square stability analysis of the trivial solution process of stochastic boundary problem of Itô-type is presented in Section 5.2. The error estimate analysis of the absolute mean square deviation of solution process of stochastic boundary value problem of Itô-type with the solution process by corresponding deterministic boundary value problem is discussed in Section 5.3. The results concerning the relative stability are outlined in Section 5.4.

5.1. GREEN'S FUNCTION METHOD

Now, we consider a boundary value problem for systems of second order stochastic differential equations of Itô-type.

We consider the stochastic boundary value problem of Itô-type (Itô-type SBVP)

$$dy' = \widehat{f}(t, y, y')dt + \sigma(t, y, y')dz(t), \qquad B_\mu y(\mu, \omega) = b_\mu(\omega) \quad (5.1.1)$$

where $\widehat{f}$ is as defined in (3.1.2); $\sigma \in C[J \times R^n \times R^n, R^{nk}]$ and $z(t) \in R[\Omega, R^k]$ is a normalized Wiener field with $z(0) = 0$, $E[z(t)] = 0$ for

$t \in J$ and

$$E[z^T(t)z(s)] = \min\{t, s\}I \qquad \text{for } t, s \in J; \tag{5.1.2}$$

dy' is a Itô-Doob stochastic differential of y'.

Before we present some results with respect to (5.1.1) in the context of Green's function approach, we consider a simple example.

Example 5.1.1. We consider the following Itô-type SBVP

$$dy' = -\Lambda\, dw, \qquad y(0) = y(1) = 0, \tag{5.1.3}$$

where Λ be constant, $y \in R$ and w is a normalized Wiener field defined on $[0,1]$ with $w(0) = 0$. From (5.1.3), we note that

$$y(t) = y(0) + \int_0^t y'(s)ds = \int_0^t y'(s)ds$$

and

$$y'(t) = y'(0) - \Lambda \int_0^t dw(s) = y'(0) - \Lambda w(t) \tag{5.1.4}$$

which imply that

$$y'(0) = \Lambda \int_0^1 w(s)ds \tag{5.1.5}$$

and hence $y'(0)$ is not independent of the increment of $w(t)$. Accordingly, Itô-Doob integral calculus is not applicable to this type of boundary value problem to obtain corresponding moment equations. From (5.1.4) and (5.1.5), we have

$$y'(t) = \Lambda \left[\int_0^1 w(s)ds - w(t)\right]. \tag{5.1.6}$$

By using (5.1.6), we can obtain the correlation function of $y'(t)$ as

$E[y'(t_1)y'(t_2)] =$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

$$\Lambda^2\left[\int_0^1\int_0^1 E[w(u)w(v)]du\,dv - \int_0^1 E[w(s)w(t_1)]ds\right.$$
$$\left. - \int_0^1 E[w(s)w(t_2)]ds + E[w(t_1)w(t_2)]\right]$$
$$= \Lambda^2\left[\frac{1}{3} - \int_0^1 \min\{s,t_1\}ds - \int_0^1 \min\{t_2,s\}ds + \min\{t_1,t_2\}\right]$$
$$= \Lambda^2\left[\frac{1}{3} - t_1\left(1-\frac{t_1}{2}\right) - t_2\left(1-\frac{t_2}{2}\right) + \min\{t_1,t_2\}\right] \quad (5.1.7)$$

and hence

$$E[y^2(t)] = \int_0^t\int_0^t E[y'(u)y'(v)]du\,dv$$
$$= \Lambda^2\int_0^t\int_0^t\left[\frac{1}{3} - u\left(1-\frac{u}{2}\right) - v\left(1-\frac{v}{2}\right) + \min\{u,v\}\right]du\,dv$$
$$= \Lambda^2\left[\frac{t^2}{3} + t\int_0^t\left(\frac{u^2}{2}-u\right)du + t\int_0^t\left(\frac{v^2}{2}-v\right)dv\right.$$
$$\left. + \int_0^t\int_0^t \min\{u,v\}du\,dv\right]$$
$$= \Lambda^2\left[\frac{t^2}{3} + t\left(\frac{t^3}{6}-\frac{t^2}{2}\right) + t\left(\frac{t^3}{6}-\frac{t^2}{2}\right)\right.$$
$$\left. + \int_0^t\int_0^v u\,du\,dv + \int_0^t\int_v^t v\,du\,dv\right]$$
$$= \Lambda^2\left[\frac{t^2}{3} + \frac{t^2}{3}(t^2-3t) + \frac{t^3}{3}\right] = \frac{\Lambda^2t^2}{3}[t^2-3t+t+1]$$
$$= \frac{\Lambda^2t^2}{3}(1-t)^2. \quad (5.1.8)$$

The above calculation of the variance of solution process $y(t)$ of (5.1.3) is simplified drastically by the use of the Green's function approach corresponding to SBVP

$$y'' = 0, \qquad y(0) = y(1) = 0 \quad (5.1.9)$$

as defined by

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1 \\ t(1-s), & 0 \le t \le s \le 1. \end{cases} \tag{5.1.10}$$

It is obvious that the process defined by

$$y(t) = \Lambda \int_0^1 G(t,s)dw(s) \tag{5.1.11}$$

which is equivalent to

$$y(t) = \Lambda \left[\int_0^t s(1-t)dw(s) + \int_t^1 t(1-s)dw(s) \right], \tag{5.1.12}$$

satisfies SBVP (5.1.3). It is obvious that the mean and variance of $y(t)$ are given by

$$E[y(t)] = 0$$

and

$$E[y^2(t)] = \int_0^1 [G(t,s)]^2 ds = \Lambda^2 \left[\int_0^t s^2(1-t)^2 ds + \int_t^1 t^2(1-s)^2 ds \right]. \tag{5.1.13}$$

Here, we have assumed that the property of independent increments of $w(t)$ at two different positions is valid. Naturally, the evaluation of (5.1.13) leads to the right-hand expression in (5.1.8). This shows one of the advantages of Green's function approach over the direct calculations. Furthermore, it is clear that the first integral in (5.1.12),

$$I_1(t) = \int_0^t s(1-t)dw(s)$$

is independent of the increment $\Delta w(t) = w(t + \Delta t) - w(t)$ of the Wiener field, i.e.,

$$E[I_1(t)\Delta w(t)] = 0.$$

But the second integral in (5.1.12) is not independent with $\Delta w(t)$. In fact

$$E\left[\left(\int_t^1 t(1-s)dw(s)\Delta w(t)\right)\right] = t(1-t)\Delta t.$$

In order to remove this dependence, we use the transformation

$$u = 1 - s \qquad \text{for } t \le s \le 1, \quad t \in [0,1]$$

and assume that

$$w(1-u) = w(1) - w(u). \tag{5.1.14}$$

With these considerations, the second integral in (5.1.12) can be rewritten as

$$I_2(t) = \int_t^1 t(1-s)dw(s) = -\int_0^{1-t} tu\, dw(1-u) = \int_0^{1-t} tu\, dw(u). \tag{5.1.15}$$

Now, it is easy to see that the integral in (5.1.15) is independent of the increment $\Delta w(t)$. Hence (5.1.14) provides a sufficient condition for the independent increment property of $w(t)$ at two end points of the interval [0,1]. Therefore (5.1.13) is justifiable.

Remark 5.1.1. We note that relation (5.1.14) is not valid with probability one. Therefore, it is not applicable to study sample properties of the Itô-type SBVP. However, it is valid in the p-th mean for $p \ge 1$. This statement can be easily verified by the coincidence of the correlation functions, i.e., for $s, t \in [0,1]$,

$$E[w(1-t)w(1-s)] = E[(w(1)-w(t))(w(1)-w(s))], \tag{5.1.16}$$

because

$$\min\{(1-t),(1-s)\} = 1 - t - s + \min\{s,t\}.$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

This suggests us that the transformed integral (5.1.15) can be useful to study statistical characteristics of SBVP such as p-th moment values or density of solution process of SBVP. Moreover, the transformed integral is independent with respect to increments of $w(t)$. Hence, we can apply the properties of Itô Doob Calculus. (5.1.14) can be considered as a transformation in the sense of (5.1.16) instead of an assumption.

A lemma analogous to Lemma 3.1.1 can be stated as follows.

Lemma 5.1.1. *Assume that*

(i) *the stochastic Green's function* $G(t,s,\omega)$ *defined in* (3.1.6) *and its derivatives*

$G_t(t,s,\omega)$ and $G_{ts}(t,s,\omega)$ satisfy

$$\max_{t\in J}\left(\int_0^1 E[|G(t,s,\omega)|^2]ds, \int_0^1 E[|G_t(t,s,\omega)|^2]ds\right) < \infty$$

and

$$\max_{t\in J}\left(E[|G_s(t,\mu,\omega)|^2], E[|G_{ts}(t,\mu,\omega)|^2]\right) < \infty;$$

(ii) *the normalized Wiener process* $z(t)$ *in* (5.1.1) *satisfies the condition* (5.1.14);

(iii) $\widehat{f}$ *and column vectors of* σ *belong to* $C[J\times R^n\times R^n, R^n]$ *and satisfy certain quadratic growth conditions;*

(iv) *for* $\mu = 0, 1$, $B_\mu(\omega)$ *and* $b_\mu(\omega)$ *are independent of* $z(t)$ *and* $z(1-t)$, $t\in[0,1]$.

Then SBVP (5.1.1) is equivalent to the stochastic integral equation

$$\begin{aligned} y(t,\omega) &= \psi(t,\omega) + \int_0^1 G(t,s,\omega)\widehat{f}(s,y(s,\omega),y'(s,\omega))ds \\ &+ \int_0^1 G(t,s,\omega)\sigma(s,y(s,\omega),y'(s,\omega))dz(s) \qquad \text{for } t\in J. \quad (5.1.17)\end{aligned}$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Proof. The proof of the lemma follows from direct application of Itô-Doob calculus [2, 31, 40, 67]. We omit the details.

Remark 5.1.2. The first and second integrals in (5.1.17) are sample Lebesgue and Itô-Doob integrals, respectively. Moreover, the second integral is defined by

$$\begin{aligned}
&\int_0^1 G(t,s,\omega)\sigma(s,y(s),y'(s))dz(s) \\
&= \int_0^t G_*(t,s,\omega)\sigma(s,y(s),y'(s))dz(s) \\
&\qquad\qquad + \int_t^1 G^*(t,s,\omega)\sigma(s,y(s),y'(s))dz(s) \\
&= \int_0^1 G_*(t,s,\omega)\sigma(s,y(s),y'(s))dz(s) \\
&\quad + \int_0^{1-t} G^*(t,1-s,\omega)\sigma(1-s,y(1-s),y'(1-s))dz(s). \quad (5.1.18)
\end{aligned}$$

We are now ready to prove an existence and uniqueness result for Itô-Doob SBVP (5.1.1) by using the contraction mapping theorem.

Theorem 5.1.1. *Let the hypotheses of* Lemma 5.1.1 *be satisfied. Furthermore, assume that*

(a) $\widehat{f}$ *and* σ *in* (5.1.1) *satisfy*

$$\|\widehat{f}(t,x,x')\|^2 + \|\sigma(t,x,x')\|^2 \le L^2(1 + \|x\|^2 + \|x'\|^2),$$

and

$$\|\widehat{f}(t,x,x') - \widehat{f}(t,y,y')\| + \|\sigma(t,x,x') - \sigma(t,y,y')\| \le L(\|x-y\| + \|x'-y'\|)$$

for some $L > 0$*;*

(b) *For* $\mu = 0, 1$, $b_\mu \in L^2[\Omega, R^n]$*;*

(c) $L\sqrt{6}\left(\max_{t\in J}\left\{\int_0^1 E[|G(t,s,\omega)|^2]ds + \int_0^1 E[|G_t(t,s,\omega)|^2]\right\}\right) < 1$, *where* L *is as defined in* (a).

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Then there exists a unique solution process of (5.1.1).

Proof. From hypotheses (ii) and (iv) of Lemma 5.1.1, we note that $G(t, s, \omega)$ is non-anticipating with respect to the non-anticipating families of sub-σ-algebra $\mathcal{F}_t$ and $\mathcal{F}_{1-t}$ relative to $z(t)$ and $z(1-t)$ for $t \in J$. From assumption (ii), the existence of solution process of (5.1.1) is equivalent to the existence of the solution process of (5.1.17). Let X be the Banach space defined by

$$X = \{x \in C^1[J, R[\Omega, R^n]] : x(t), x'(t) \text{ are } \mathcal{F}_t\text{-measurable and } x, x' \in L^2[J \times \Omega, R^n]\}$$

with the norm $\|x\|_X = \|x\|^0_{L^2[\Omega,R^n]} + \|x'\|^0_{L^2[\Omega,R^n]}$, where

$$\|x\|^0_{L^2[\Omega,R^n]} = \max_{t\in J}[E[\|x(t)\|^2]]^{1/2}$$
$$\|x'\|^0_{L^2[\Omega,R^n]} = \max_{t\in J}[E[\|x'(t)\|^2]]^{1/2},$$

and $x'(t)$ stands for the sample derivative of $x(t)$. We define an operator T on X as follows

$$(Tx)(t) = \psi(t, \omega) + \int_0^1 G(t, s, \omega)\widehat{f}(s, x(s), x'(s))ds + \int_0^1 G(t, s, \omega)\sigma(s, x(s), x'(s))dz(s). \quad (5.1.19)$$

From the assumptions Lemma 5.1.1, the first and second integrals in (5.1.19) are well-defined in the sense of sample Lebesgue and Itô-Doob. The $\mathcal{F}_t$-measurability of $(Tx)(t)$ follows from $\mathcal{F}_t$-measurability of $\psi(t, \omega)$ and $x(t)$. The first sample derivative of $(Tx)(t)$ is given by

$$(Tx)'(t) = \psi'(t, \omega) + \int_0^1 G_t(t, s, \omega)\widehat{f}(s, x(s), x'(s))ds + \int_0^1 G_t(t, s, \omega)\sigma(s, x(s), x'(s))dz(s). \quad (5.1.20)$$

Because of the smoothness of $G(t,s,\omega)$ together with assumptions on $G(t,s,\omega)$ and $z(t)$, the above derivation is possible. Again, from the assumptions of Lemma 5.1.1 and condition (b), the $\mathcal{F}_t$-measurability of $(Tx)'(t)$ follows, immediately. Since $\left(\sum_{i=1}^{4} a_i\right)^2 \leq 4\left(\sum_{i=1}^{4} a_i^2\right)$, the use of (5.1.17), (5.1.19), Hölder's inequality, assumptions of Lemma 5.1.1, Remark 5.1.2 and conditions (a) and (b), we have

$$
\begin{aligned}
&E[\|(Tx)(t)\|^2] \\
&\leq 4\Big\{E\|\psi(t,\omega)\|^2] + E\left[\left\|\int_0^1 G(t,s,\omega)\widehat{f}(s,x(s),x'(s))ds\right\|^2\right] \\
&\qquad + \int_0^t E[|G_*(t,s,\omega)|^2]E[\|\sigma(s,x(s),x'(s))\|^2]ds \\
&\qquad + \int_0^{1-t} E[|G^*(t,s,\omega)|^2]E[\|\sigma(s,x(s),x'(s))\|^2]ds\Big\} \\
&\leq 4\Big\{E[\|\psi(t,\omega)\|^2] + \int_0^1 E[|G(t,s,\omega)|^2]E[\|\widehat{f}(s,x(s),x'(s))\|^2]ds \\
&\qquad + \int_0^1 E[|G(t,s,\omega)|^2]E[\|\sigma(s,x(s),x'(s))\|^2]ds\Big\} \\
&\leq 4\Big\{E[\|\psi(t,\omega\|^2] \\
&\qquad +2L^2\int_0^1 E[|G(t,s,\omega)|^2](1+E[\|x(s)\|^2]+E[\|x'(s)\|^2])ds\Big\}.
\end{aligned}
$$

Therefore, one can conclude that $\|Tx\|_{L^2[\Omega,R^n]}$ is finite. To conclude $Tx \in X$, we need to show that $\|(Tx)'\|_{L^2[\Omega,R^n]}$ is finite. By following the above argument and using (5.1.20), one can easily show that

$$
\begin{aligned}
E[\|(Tx)'(t)\|^2] &\leq 4[E[\|\psi'(t,\omega)\|^2] \\
&\qquad +2L^2\int_0^1 E[|G_t(t,s,\omega)|^2](1+E[\|x(s)\|^2]+E[\|x'(s)\|^2])ds]
\end{aligned}
$$

and it is finite. Thus verifying the fact that T maps X into itself. Now, we show that T is continuous on X. By considering, $x, y \in X$,

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

one can obtain

$$
\begin{aligned}
&E[\|(Tx)(t) - (Ty)(t)\|^2] \\
&\le 3\Bigg[E\left[\int_0^1 |G(t,s,\omega)|\|\widehat{f}(s,x(s),x'(s)) - \widehat{f}(s,x(s),y'(s))\|ds\right]^2 \\
&\qquad + \int_0^t E[|G_*(t,s,\omega)|^2]E[\|\sigma(s,x(s),x'(s)) - \sigma(s,y(s),y'(s))\|^2]ds \\
&\qquad\quad + \int_0^{1-t} E[|G^*(t,1-s,\omega)|^2]E[\|\sigma(1-s,x(1-s),x'(1-s)) \\
&\qquad\qquad\qquad -\sigma(1-s,y(1-s),y'(1-s))\|^2]ds\Bigg] \\
&\le 6L^2\Bigg\{\int_0^1 E[|G(t,s,\omega)|^2](E[\|x(s)-y(s)\|^2 \\
&\qquad\qquad\qquad +E[\|x'(s)-y'(s)\|^2])ds\Bigg\} \\
&\le 6L^2\left(\int_0^1 E[|G(t,s,\omega)|^2]\right)\|x-y\|_X^2.
\end{aligned}
$$

Similarly, one can obtain

$$
E[\|(Tx)'(t) - (Ty)'(t)\|^2] \le 6L^2\left(\int_0^1 E[|G_t(t,s,\omega)|^2]\right)\|x-y\|_X^2.
$$

From these considerations, one establishes

$$
\begin{aligned}
\|Tx - Ty\|_X \le L\sqrt{6}\Bigg(&\left[\int_0^1 E[|G(t,s,\omega)|^2]ds\right]^{1/2} \\
&+ \left[\int_0^1 E[|G_t(t,s,\omega)|^2]ds\right]^{1/2}\Bigg)\|x-y\|_X. \quad (5.1.21)
\end{aligned}
$$

From this and (c), we conclude that T is continuous on X. Moreover, it is a contraction mapping on X. Hence, there exists a fixed point of T. This fixed point is a solution process of (5.1.20). Then for Lemma 5.1.1, we conclude that this fixed point is also the unique solution process of (5.1.1). This completes the proof of the theorem.

5.2. STABILITY ANALYSIS

By employing the integral representation of a solution process of (SBVP) (5.1.1), sufficient conditions are given for the mean-square stability of the trivial solution process of (5.1.1) with boundary conditions as described in (H_2'').

Theorem 5.2.1. *Assume that*

(i) *the hypotheses of* Lemma 5.1.1 *are satisfied;*

(ii) $\widehat{f}(t,0,0) \equiv 0$ *and* $\sigma(t,0,0) \equiv 0$ *for all* $t \in [0,1]$*;*

(iii) $\|\widehat{f}(t,u,v)\|^2 + \|\sigma(t,u,v)\|^2 \leq k_1(t)\|u\|^2 + k_2(t)\|v\|^2$ *where* $k_1, k_2 \in C[[0,1], R_+]$*;*

(iv) *a linear operator* $\bar{H}$ *defined on* $C[[0,1],R] \times C[[0,1],R]$ *into itself by*

$$\bar{H}(u,v)(t) = \begin{bmatrix} 2^4 \int_0^1 E[\|G(t,s,\omega)\|^2](k_1(s)u(s) + k_2(s)v(s))ds \\ 2^4 \int_0^1 E[\|G_t(t,s,\omega)\|^2](k_1(s)u(s) + k_2(s)v(s))ds \end{bmatrix} \tag{5.2.1}$$

has a spectral radius less than 1, *where* k_1 *and* k_2 *as defined in* (iii) *and* G *is the Green's function defined in* (3.1.6). *Then the trivial solution process of* (5.1.1) *is mean-square stable.*

Proof. From (5.1.17), for $t \in [0,1]$ we obtain

$$\|y(t,\omega)\|^2 \leq 4\|\psi(t,\omega)\|^2 + 8\left\|\int_0^1 G(t,s,\omega)\widehat{f}(s,y(s,\omega),y'(s,\omega))ds\right\|^2$$
$$+8\left\|\int_0^1 G(t,s,\omega)\sigma(s,y(s,\omega),y'(s,\omega))dz(s)\right\|^2.$$

By taking expectation both sides of the above inequality and using the properties of Itô-Doob calculus, we have

$$E[\|y(t,\omega)\|^2] \leq 4E[\|\psi(t,\omega)\|^2]$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

$$+8E\left[\left\|\int_0^1 G(t,s,\omega)\widehat{f}(s,y(s,\omega),y'(s,\omega))ds\right\|^2\right]$$
$$+8\int_0^1 E[\|G(t,s,\omega)\sigma(s,y(s,\omega),y'(s,\omega))\|^2]ds.$$

This together with assumptions (i), (iii) and Hölder inequality gives rise to

$$\begin{aligned}
&E[\|y(t,\omega)\|^2] \leq 4E[\|\psi(t,\omega)\|^2] \\
&\quad +8\int_0^1 E[\|G(t,s,\omega)\|^2](k_1(s)E[\|y(s,\omega)\|^2]+k_2(s)E[\|y'(s,\omega)\|^2])ds \\
&\quad +8\int_0^1 E[\|G(t,s,\omega)\|^2](k_1(s)E[\|y(s,\omega)\|^2]+k_2(s)E[\|y'(s,\omega)\|^2])ds \\
&\leq 4E[\|\psi(t,\omega)\|^2] \\
&+16\int_0^1 E[\|G(t,s,\omega)\|^2]\left(k_1(s)E[\|y(s,\omega)\|^2]+k_2(s)E[\|y'(s,\omega)\|^2]\right)ds.
\end{aligned} \tag{5.2.2}$$

From (5.1.17), it is easy to see that for $t \in [0,1]$,

$$\begin{aligned}
y'(t,\omega) = \psi'(t,\omega) &+ \int_0^1 G_t(t,s,\omega)\widehat{f}(s,y(s,\omega),y'(s,\omega))ds \\
&+ \int_0^1 G_t(t,s,\omega)\sigma(s,y(s,\omega),y'(s,\omega))dz(s).
\end{aligned} \tag{5.2.3}$$

From (5.2.3) and imitating the above argument, we obtain

$$\begin{aligned}
&E[\|y'(t,\omega)\|^2] \leq 4E[\|\psi'(t,\omega)\|^2] \\
&+16\int_0^1 E[\|G_t(t,s,\omega)\|^2]\left(k_1(s)E[\|y(s,\omega)\|^2]+k_2(s)E[\|y'(s,\omega)\|^2]\right)ds.
\end{aligned} \tag{5.2.4}$$

From (5.2.2), (5.2.4) and the definition of $\bar{H}$ in (5.2.1), for $t \in [0,1]$ we have

$$\begin{aligned}
&(E[\|y(t,\omega)\|^2], E[\|y'(t,\omega)\|^2]) \leq \\
&\qquad \bar{H}(E[\|y(\cdot,\omega)\|^2, E[\|y'(\cdot,\omega)\|^2])(t) + (M_0, M_1)(t),
\end{aligned} \tag{5.2.5}$$

where

$$M_0(t) = 4E[\|\psi(t,\omega)\|^2] \qquad \text{and} \qquad M_1(t) = 4E[\|\psi'(t,\omega)\|^2].$$

Hence

$$(E[\|y(t,\omega)\|]^2, E[\|y'(t,\omega)\|^2]) \leq$$
$$(I - \bar{H})^{-1}(M_0, M_1)(t) \leq \sum_{m=1}^{\infty} \bar{H}^m (M_0, M_1)(t).$$

This implies that

$$E[\|y(t,\omega)\|^2] + E[\|y'(t,\omega)\|^2] \leq$$
$$(1 - \|\bar{H}\|)^{-1}(\|M_0\|_0 + \|M_1\|_0) \qquad \text{for } t \in [0,1], \quad (5.2.6)$$

where

$$\|M_\mu\|_0 = \max_{t\in[0,1]} |M_\mu(t)| \qquad \text{for } \mu = 0, 1.$$

Thus

$$E[\|y(t,\omega)\|^2] \leq (1 - \|\bar{H}\|)^{-1}(\|M_0\|_0 + \|M_1\|_0) \qquad \text{for } t \in [0,1]. \quad (5.2.7)$$

For any $\epsilon > 0$, we can choose $b_\mu(\omega)$ for $\mu = 0, 1$, so that

$$(1 - \|\bar{H}\|)^{-1}(\|M_0\|_0 + \|M_1\|_0) < \epsilon^2. \qquad (5.2.8)$$

This and (5.2.7), we obtain

$$(E[\|y(t,\omega)\|^2])^{1/2} < \epsilon \qquad \text{for } t \in [0,1]$$

whenever b_μ's satisfy (5.2.8). This completes the proof of the theorem.

Remark 5.2.1. Although Theorem 5.2.1 gives sufficient conditions for the mean-square stability of the trivial solution process of (5.1.1), the sufficient conditions for the $2p$-th moment stability of the trivial solutions process can be given, analogously. For this purpose, we need to use the following property of the Itô-Doob integral:

$$E\left[\left\|\int_0^1 F(s)dz(s)\right\|^{2p}\right] \le (p(2p-1))^p \int_0^1 E[\|F(s)\|^{2p}]ds, \quad (5.2.9)$$

where F is nonanticipating $n \times k$ matrix process relative to $\mathcal{F}_t$-sub-σ-algebra of $\mathcal{F}$ and F satisfies $\int_0^1 E[\|F(s)\|^{2p}] < \infty$ and p is a positive integer.

5.3. ERROR ESTIMATES

In this section, we use the Green's function representation (5.1.17) of solution process of (5.1.1) to analyze the absolute mean-square deviation of solution of corresponding smooth deterministic boundary value problem (DBVP). For this purpose, we need to present the following preliminary material.

We consider the stochastic boundary value problem (SBVP) (5.1.1) and its corresponding deterministic boundary value problem (DBVP)

$$m'' = \widehat{f}(t, m, m'), \qquad \widehat{B}_\mu m(\mu) = \widehat{b}_\mu \quad (5.3.1)$$

which is obtained by ignoring the randomness in the system. In our presentation we will be using the following boundary value problem

$$x'' = \widehat{f}(t, x, x'), \qquad B_\mu(\omega)x(\mu, \omega) = b_\mu(\omega) \qquad \text{for } \mu = 0, 1 \quad (5.3.2)$$

with stochastic boundary conditions. In (5.3.1) and (5.3.2), $m, x \in R^n$; boundary operators $B_\mu(\omega)$ and $\widehat{B}_\mu$ with $b_\mu(\omega)$ and $\widehat{b}_\mu$ for $\mu = 0, 1$

are as defined in (3.1.3) and (3.1.4); $\widehat{f} \in C[J \times R^n \times R^n, R^n]$. Let $G(t,s,\omega)$ and $\widehat{G}(t,s)$ be stochastic and deterministic Green's functions as defined in (3.1.6) and (3.6.3), respectively.

Now, we are ready to present a result concerning the absolute mean-square deviation of solution process of (5.1.1) with either (5.3.1) or (5.3.2). Let $y(t) = y(t,\omega)$ be a solution process of SBVP (5.1.1) and let $m(t)$ be a solution of (5.3.1).

Theorem 5.3.1. *Assume that*

(i) *the hypotheses of* Lemma 5.1.1 *are satisfied;*

(ii) $\widehat{f} \in C[J \times R^n \times R^n, R^n]$ *and* $\sigma \in C[J \times R^n \times R^n, R^{n\times k}]$ *and satisfy*

$$\begin{cases} \|\widehat{f}(t,x,y) - \widehat{f}(t,u,v)\|^2 + \|\sigma(t,x,y) - \sigma(t,u,v)\|^2 \le \\ \qquad\qquad k_1(t)\|x-u\|^2 + k_2(t)\|y-v\|^2 \\ \|\widehat{f}(t,u,v)\|^2 + \|\sigma(t,u,v)\|^2 \le k_3(t)\|u\|^2 + k_4(t)\|v\|^2 + k_5(t) \end{cases}$$

for $(t,x,y),(t,u,v) \in J \times R^n \times R^n$, *where* $k_i \in C[J, R_+]$ *for* $i = 1,2,\ldots,5$;

(iii) *a linear operator* $\bar{h}$ *defined on* $C[J,R] \times C[J,R]$ *into itself by*

$$\bar{H}(u,v)(t) = \begin{bmatrix} \int_0^1 E[\|G(t,s,\omega)\|^2](k_1(s)u(s) + k_2(s)v(s))ds \\ \int_0^1 E[\|G_t(t,s,\omega)\|^2](k_1(s)u(s) + k_2(s)v(s))ds \end{bmatrix},$$

has a spectral radius less than one. Then

$$(E[\|y(t) - m(t)\|^2], E[\|y'(t) - m'(t)\|^2]) \le \sum_{m=0}^{\infty} (\bar{H})^m (M_0, M_1)(t) \qquad \text{for } t \in J. \quad (5.3.3)$$

Moreover,

$$(E[\|y(t) - m(t)\|^2], E[\|y'(t) - m'(t)\|^2]) \le \left(\sum_{m=0}^{\infty} \|\bar{H}\|^m\right)(\|M_0\|_0, \|M_1\|_0)(t) \qquad \text{for } t \in J, \quad (5.3.4)$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

where

$$M_0(t) = 4E[\|\psi(t,\omega) - \widehat{\psi}(t)\|^2]$$
$$+2^8 \int_0^1 (E[\|G(t,s,\omega) - \widehat{G}(t,s)\|^2(\|\widehat{f}(s,m(s),m'(s))\|^2$$
$$+\|\sigma(s,m(s),m'(s))\|^2))ds + 2^7 \int_0^1 \|\widehat{G}(t,s)\|^2\|\sigma(s,m(s),m'(s))\|^2 ds$$
$$M_1(t) = 4E[\|\psi'(t,\omega) - \widehat{\psi}(t)\|^2]$$
$$+2^8 \int_0^1 (E[\|G_t(t,s,\omega) - \widehat{G}_t(t,s)\|^2(\|\widehat{f}(s,m(s),m'(s))\|^2$$
$$+\|\sigma(s,m(s),m'(s))\|^2))ds$$
$$+2^7 \int_0^1 \|\widehat{G}_t(t,s)\|^2\|\sigma(s,m(s),m'(s))\|^2 ds \quad (5.3.5)$$

and

$$\|M_\mu\|_0 = \max_{t\in J}[M_\mu(t)] \qquad \text{for } \mu = 0, 1.$$

Proof. From (5.1.17) and (3.6.3), we write

$$y(t,\omega) - m(t) = \psi(t,\omega) - \widehat{\psi}(t)$$
$$+ \int_0^1 G(t,s,\omega)(\widehat{f}(s,y(s,\omega),y'(s,\omega)) - \widehat{f}(s,m(s),m'(s)))ds$$
$$+ \int_0^1 (G(t,s,\omega) - \widehat{G}(t,s,\omega))\widehat{f}(s,m(s),m'(s))ds$$
$$+ \int_0^1 G(t,s,\omega)(\sigma(s,y(s,\omega),y'(s,\omega)) - \sigma(s,m(s),m'(s)))dz(s)$$
$$+ \int_0^1 (G(t,s,\omega) - \widehat{G}(t,s))\sigma(s,m(s),m'(s))dz(s)$$
$$+ \int_0^1 \widehat{G}(t,s)\sigma(s,m(s),m'(s))dz(s).$$

By taking norm and squaring both sides of the above expression, using $\left[\sum_{i=1}^n a_i\right]^2 \le 2^n \left(\sum_{i=1}^n a_i^2\right)$ and Cauchy-Schwartz inequality we obtain

$$\|y(t,\omega) - m(t)\|^2 \le 4[\|\psi(t,\omega) - \widehat{\psi}(t)\|^2]$$

$$+2^7\Bigg\{\int_0^1 \|G(t,s,\omega)\|^2(\|\widehat{f}(s,y,y') - \widehat{f}(s,m,m')\|^2)ds$$

$$+\int_0^1 \|G(t,s,\omega) - \widehat{G}(t,s)\|^2(\|\widehat{f}(s,m,m')\|^2)ds$$

$$+\left\|\int_0^1 G(t,s,\omega)(\sigma(s,y(s,\omega),y'(s,\omega)) - \sigma(s,m(s),m'(s)))dz(s)\right\|^2$$

$$+\left\|\int_0^1 (G(t,s,\omega) - \widehat{G}(t,s))(\sigma(s,m(s),m'(s)))dz(s)\right\|^2$$

$$+\left\|\widehat{G}(t,s)\sigma(s,m(s),m'(s))dz(s)\right\|^2\Bigg\}. \quad (5.3.6)$$

This together with Itô-Doob integral calculus and hypothesis (ii) yields

$$E[\|y(t,\omega) - m(t)\|^2] \leq 4\Bigg\{E[\|\psi(t,\omega) - \widehat{\psi}(t)\|^2]$$

$$+2^7\Bigg\{\int_0^1 E[\|G(t,s,\omega)\|^2](k_1(s)E[\|y(s,\omega) - m(s)\|^2]$$

$$+k_2(s)E[\|y'(s,\omega) - m'(s)\|^2])ds$$

$$+\int_0^1 E[\|G(t,s,\omega) - \widehat{G}(t,s)\|^2]\|\widehat{f}(s,m(s),m'(s))\|^2 ds$$

$$+\int_0^1 E[\|G(t,s,\omega)\|^2](k_1(s)E[\|y(s,\omega) - m(s)\|^2]$$

$$+k_2(s)E[\|y'(s,\omega) - m'(s)\|^2])ds$$

$$+\int_0^1 E[\|G(t,s,\omega) - \widehat{G}(t,s)\|^2]\|\sigma(s,m(s),m'(s))\|^2 ds$$

$$+\int_0^1 \|\widehat{G}(t,s)\|^2\|\sigma(s,m(s),m'(s))\|^2 ds\Bigg\}\Bigg\}.$$

From the definition of $M_0(t)$ and the above inequality, we get

$$E[\|y(t) - m(t)\|^2] \leq$$

$$M_0(t) + 2^8\int_0^1 E[\|G(t,s,\omega)\|^2](k_1(s)E[\|y(s) - m(s)\|^2]$$

$$+k_2(s)E[\|y'(s) - m'(s)\|^2])ds. \quad (5.3.7)$$

Similarly, from (5.1.17) and (3.6.3) we have

$$
\begin{aligned}
y'(t) - m'(t) = {} & \psi'(t,\omega) - \widehat{\psi}'(t) \\
& + \int_0^1 G_t(s,t,\omega)(\widehat{f}(s,y(s,\omega),y'(s,\omega) - \widehat{f}(s,m(s),m'(s))))ds \\
& + \int_0^1 (G_t(t,s,\omega) - \widehat{G}_t(t,s))\widehat{f}(s,m(s),m'(s))ds \\
& + \int_0^1 G_t(t,s,\omega)(\sigma(s,y(s,\omega),y'(s,\omega)) - \sigma(s,m(s),m'(s)))dz(s) \\
& + \int_0^1 (G_t(t,s,\omega) - \widehat{G}_t(t,s))\sigma(s,m(s),m'(s))dz(s) \\
& + \int_0^1 \widehat{G}_t(t,s)\sigma(s,m(s),m'(s))dz(s). \qquad (5.3.8)
\end{aligned}
$$

By repeating the arguments used in deriving the inequalities (5.3.6) and (5.3.7) and using (5.3.8) and definition of $M_1(t)$, we obtain

$$
\begin{aligned}
& E[\|y'(t) - m'(t)\|^2] \leq \\
& \qquad M_1(t) + 2^8 \int_0^1 E[\|G_t(t,s,\omega)\|^2](k_1(s)E[\|y(s) - m(s)\|^2] \\
& \qquad\qquad + k_2(s)E[\|y'(s) - m'(s)\|^2]ds. \qquad (5.3.9)
\end{aligned}
$$

From (5.3.7), (5.3.9) and the definition of $\bar{H}$ in assumption (iii), we have

$$
\begin{aligned}
& (E[\|y(t) - m(t)\|^2], E[\|y'(t) - m(t)\|^2]) \leq \\
& \qquad \bar{H}(E[\|y(\cdot,\omega) - m(\cdot)\|^2, E[\|y'(\cdot,\omega) - m'(\cdot)\|^2])(t) + (M_0, M_1)(t).
\end{aligned}
$$

This together with assumption (iii) yields

$$
\begin{aligned}
& (E[\|y(t) - m(t)\|^2], E[\|y'(t) - m'(t)\|^2]) \leq \\
& \qquad (I - \bar{H})^{-1}(M_0, M_1)(t) \leq \sum_{m=0}^{\infty} (\bar{H})^m (M_0, M_1)(t) \qquad \text{for } t \in J.
\end{aligned}
$$

This establishes the validity of (5.3.3). Moreover, (5.3.4) follows from the fact that $\|\bar{H}\| < 1$. This completes the proof of the theorem.

Remark 5.3.1. For the special values of $\alpha_0(\omega) = \alpha_1(\omega) = 1$ and $\beta_0(\omega) = \beta_1(\omega) = 0$, it is obvious that $\widehat{\alpha}_0 = \widehat{\alpha}_1 = 1$ and $\widehat{\beta}_0 = \widehat{\beta}_1 = 0$. Thus $G(t,s,\omega) = \widehat{G}(t,s)$ which is defined in Remark 3.6.1. In this case, M_0 and M_1 in (5.3.3) reduce to

$$\begin{aligned} M_0(t) &= 4E[\|\psi(t,\omega) - \widehat{\psi}(t)\|^2] \\ &\qquad +2^7 \int_0^1 \|\widehat{G}(t,s)\|^2 \|\sigma(s, m(s), m'(s))\|^2 ds \\ M_1(t) &= 4E[\|\psi'(t) - \widehat{\psi}(t)\|^2] \\ &\qquad +2^7 \int_0^1 \|\widehat{G}_t(t,s)\|^2 \|\sigma(s, m(s), m'(s))\|^2 ds, \end{aligned} \quad (5.3.10)$$

where

$$\psi(t,\omega) = b_0(\omega)(1-t) + b_1(\omega)t, \qquad \widehat{\psi}(t) = \widehat{b}_0(1-t) + \widehat{b}_1 t$$

and

$$\psi'(t,\omega) = -b_0(\omega) + b_1(\omega), \qquad \widehat{\psi}'(t) = -\widehat{b}_0 + \widehat{b}_1.$$

Remark 5.3.2. Although Theorem 5.3.1 gives absolute mean-square error estimates between SBVP (5.1.1) and (5.3.1), the results can be established $2p$-th moment error estimates for $p \geq 1$. For this purpose, we need the following property of Itô-Doob integral

$$E\left[\left\|\int_0^1 F(s)dz(s)\right\|^{2p}\right] \leq (p(2p-1)) \int_0^1 E[\|F(s)\|^{2p}]ds, \quad (5.3.11)$$

where F is nonanticipating $n \times k$ matrix process relative to $\mathcal{F}_t$ with $\int_0^1 E[\|F(s)\|^{2p}]ds < \infty$ and p is a positive integer.

5.4. RELATIVE STABILITY

In this section, we establish the relative stability property of (5.1.1) relative to (5.3.1). The following result gives sufficient conditions for the relative stability of (5.1.1) with respect to (5.3.1).

Theorem 5.4.1. *Let the hypotheses of* Theorem 5.3.1 *be satisfied. Further assume the* $\|M_\mu\|_0$ *can be made arbitrarily small for* $\mu = 0, 1$. *Then the system* (5.1.1) *and* (5.3.1) *are relatively stable in second moment.*

Proof. By following the proof of Theorem 5.3.1, we arrive at

$$E[\|y(t)-m(t)\|^2+\|y'(t)-m'(t)\|^2] \leq \left(\sum_{m=0}^{\infty} \|\bar{H}\|^m\right)(\|M_0\|_0+\|M_1\|_0). \tag{5.4.1}$$

Since $\|\bar{H}\| < 1$, therefore $\sum_{m=0}^{\infty} \|\bar{H}\|^m = (1 - \|\bar{H}\|)^{-1}$. This together with (5.4.1), we have

$$E[\|y(t) - m(t)\|^2] \leq (1 - \|\bar{H}\|)^{-1}(\|M_0\|_0 + \|M_1\|_0). \tag{5.4.2}$$

From the hypothesis that $\|M_0\|_0$ and $\|M_1\|_0$ can be made arbitrarily small, therefore, for every $\epsilon > 0$, one can choose $\delta(\epsilon) > 0$ such that

$$E[\|y(t) - m(t)\|^2] < \epsilon^2 \qquad \text{for } t \in [0, 1]$$

whenever

$$(\|M_0\|_0 + \|M_1\|_0) < \epsilon^2(1 - \|\bar{H}\|).$$

This implies that

$$\|y(t) - m(t)\|_2 < \epsilon \qquad \text{for } t \in [0, 1]$$

whenever $E[\|\psi(t) - \widehat{\psi}(t)\|^2] < \frac{1}{2}\,\epsilon^2(1 - \|\bar{H}\|)$ and the second and the third terms in M_0 and M_1 can be made less than $\frac{1}{2}\,\epsilon^2(1 - \|\bar{H}\|)$. This proves the theorem.

Remark 5.4.1. A remark similar to Remark 5.3.1 can be made, analogously. Further details are left to the reader.

Remark 5.4.2. We note that the assumption about the arbitrarily smallness of $\|M_0\|_0$ and $\|M_1\|_0$ seems to be stringent. The feasibility of such an assumption is that the DBVP (5.3.1) must be stable. In the context of this remark, the concept of relative stability of (5.1.1) with respect to (5.3.1) can be termed as the relative stability under perturbations.

5.5. NOTES AND COMMENTS

A formulation of stochastic boundary value problem of Itô-type is based on the ideas of Weding [110]. Example 5.1.1 is adapted from Weding [110] and it initiates a formulation of SBVP of Itô-type. Lemma 5.1.1 and Theorem 5.1.1 are taken from Ladde [63]. The results of Sections 5.2, 5.3, and 5.4 are new and they are based on results of Sections 3.4, 3.5, 3.6, and 3.7 of Chapter 3.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

APPENDIX

A.0. INTRODUCTION

The main objective of the appendices is to provide an easy reference to concepts, preliminary results and definitions that are used in the text. Section A.1 deals with the strong laws of large numbers. In Section A.2 several fundamental results in the study of initial value problems are given. Finally, in Section A.3, the basic results in boundary value problems are stated.

A.1. CONVERGENCE OF RANDOM SEQUENCES

Let X_n be a sequence of real valued random variables adapted to an increasing sequence of σ-algebras $\mathcal{F}_n$. We denote by $\bar{T}$ the collection of arbitrary stopping times for $(\mathcal{F}_n)_{n\in N}$.

Theorem A.1.1. [54] *For independent random variable X_n and for processes of the form $X_n = c_n^{-1}\sum_{i=1}^n Y_i$ with increasing c_n's and independent nonnegative Y_i's*

$$E[\sup X_n^+] \leq 2 \sup_{\mu\in\bar{T}} E[X_\mu].$$

Let α, β, and δ be three arbitrary numbers such that $0 \leq \alpha < \beta \leq 2\pi$, and $\delta \in (0,1]$. Consider the following subsets of the complex plane $\mathbf{Z}$:

$$B = \{z : \alpha < \arg z < \beta\}$$

$$C = \{z : 1-\delta \leq |z| \leq 1+\delta\}.$$

Theorem A.1.2. Šparo and Šur [105] *Let the coefficients $a_k(\omega)$ of a random algebraic polynomial $P_n(z,\omega)$ be independent and identically distributed complex-valued random variables, and let $E[\log^+|a_k|] < \infty$, $k = 0,1,\ldots,n$, and $\log^+|a_k| = \max\{0,\log|a_k|\}$. Then*

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

(a) $\lim_{n\to\infty} n^{-1}N_n(C,\omega) = 1$, *and*

(b) $\lim_{n\to\infty} n^{-1}N_n(B,\omega) = (\beta - \alpha)/2\pi$ *in probability.*

The following well-known inequality gives an important relationship concerning convex functions and the expected value of random variables.

Theorem A.1.3. Jensen's Inequality [39] *Suppose that $\phi : G \to R$ is a convex function defined on an open subinterval G of R and that X is a random variable such that $E(|X|) < \infty$, $P(X \in G) = 1$, and $E|\phi(X)| < \infty$. Then*

$$E[\phi(X)] \geq \phi(E[X]).$$

Theorem A.1.4. Chebyshev's Inequality [39] *Let X be a R^n valued random vector defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Then for every $\epsilon > 0$, $p \geq 1$, and $X \in L^p[(\Omega, \mathcal{F}, P), R^n]$,*

$$P(\{\omega \in \Omega : \|x(\omega)\| > \epsilon\}) \leq \frac{E[\|X\|^p]}{\epsilon^p}.$$

Theorem A.1.5. [31] *If $(X(t), \mathcal{F}_t)$ is non-negative separable super martingale, then the finite limit*

$$\lim_{t\to\infty} x(t,\omega) = x(\omega)$$

exists with probability one.

Theorem A.1.6. Hölder Inequality [67, 98] *Let $f \in L^p[(\Omega, \mathcal{F}, P), R^n]$ and $g \in L^q[(\Omega, \mathcal{F}, P), R^n]$ if p and q are real numbers greater than 1 such that $\frac{1}{p} + \frac{1}{q} = 1$, then $fg \in L[(\Omega, \mathcal{F}, P), R^n]$ and*

$$\|fg\| \leq \|f\|_p \|g\|_q.$$

Theorem A.1.7. Lindeberg Central Limit Theorem [37] *Let $\{X_n\}$ be a sequence of real valued mutually independent random variables with $E[X_n] = \mu_n$, $E[(X_n - \mu_n)^2] = \sigma_n^2$, and assume*

$$\lim_{n\to\infty}\left[V_n^{-1}\sum_{i=1}^{n}\int_{|z-\mu_i|>\epsilon V_n^{1/2}}(E-\mu_i)^2 dF_i(z)\right] = 0,$$

$\forall\ \epsilon > 0$, where $V_n = \sum_{i=1}^{n}\sigma_n^2$. Then

$$\sum_{i=1}^{n}(X_i - \mu_i)/V_n^{1/2} \to N(0,1),$$

in the sense of distribution as $n \to \infty$.

Theorem A.1.8. Strong Law of Large Numbers [27] *Let $\{X_n\}$ be a sequence of real valued mutually independent random variables with $E[X_n] = 0$ and $\sum_{i=1}^{\infty}\left(\frac{\sigma_i^2}{i^2}\right)$ finite. Then*

$$\lim_{n\to\infty}\left[\frac{\sum_{i=1}^{n}X_i}{n}\right] = 0 \text{ a.s.}$$

A.2. INITIAL VALUE PROBLEMS

We present some of the most important concepts and results in theory of systems of stochastic differential inequalities. For more details, see Ladde and Lakshmikantham [67].

Theorem A.2.1. [67] *Assume that $\widehat{F}$ in (2.1.3) belongs to $C[R_+ \times R^n, R^n]$ and possesses continuous partial derivatives $\frac{\partial \widehat{F}}{\partial y}$ on $R_+ \times R^n$. Let the sample solution process $x(t, t_0, x_0(\omega))$ of (2.1.3) exist for $t \geq t_0$. Then*

(a) *the sample derivative $\frac{\partial x}{\partial x_0}(t, t_0, x_0(\omega)) = \Phi(t, t_0, x_0(\omega))$ exists and is the solution process of*

$$w' = \widehat{F}_x(t, x(t, t_0, x_0(\omega)))w \tag{A.2.1}$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

such that $\Phi(t_0, t_0, x_0(\omega))$ *is the unit matrix;*

(b) *the sample derivative* $\frac{\partial x}{\partial t_0}(t, t_0, x_0(\omega))$ *exists and satisfies* (A.2.1) *with*

$$\frac{\partial x}{\partial t_0}(t, t_0, x_0(\omega)) = -\Phi(t, t_0, x_0(\omega))\widehat{F}(t_0, x_0(\omega)), \qquad t \geq t_0. \tag{A.2.2}$$

The following lemma is a stochastic version of generalized mean-value theorem.

Lemma A.2.1. **[67]** *Assume that* $F \in M[R_+ \times R^n, R[\Omega, R^n]]$ *and its sample derivative*

$$\frac{\partial F}{\partial y}(t, y, \omega) = F_y(t, y, \omega)$$

exists and is sample continuous in y *for each* $t \in R_+$. *Then*

$$F(t, y, \omega) - F(t, x, \omega) = \int_0^1 \frac{\partial F}{\partial y}(t, sy(1-s)x, \omega)(y - x)ds. \tag{A.2.3}$$

The following lemma gives a relationship between the fundamental solution of variational system (A.2.1) and solutions of (2.1.2) and (2.1.3).

Lemma A.2.2. **[67]** *Assume that the hypotheses of* Theorem A.2.1 *hold. Furthermore, assume that* $x(t, t_0, m_0) = m(t)$ *and* $x(t, t_0, x_0(\omega)) = x(t, \omega)$ *are solutions of* (2.1.2) *and* (2.1.3) *existing for* $t \geq t_0$. *Then for* $t \geq t_0$

$$x(t, \omega) - m(t) = \int_0^1 \Phi(t, m_0 + s(x_0(\omega) - m_0))ds(x_0(\omega) - m_0). \tag{A.2.4}$$

Theorem A.2.2. **[43]** *Assume that* $\widehat{F}$ *in* (2.1.3) *belong to* $C[R_+ \times R^n, R^n]$ *and possesses* m *order derivatives for* $m \geq 1$ *with respect*

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

to y for each $t \in R_+$. Then $x(t, t_0, x_0)$ has all continuous partial derivatives of the form

$$\frac{\partial^{i+i_0+j_1+j_2+\cdots+j_n} x(t, t_0, x_0)}{\partial t^i \partial t_0^{i_0} (\partial y_{01})^{j_1} (\partial y_{02})^{j_2} \cdots (\partial y_{0n})^{j_n}} \tag{A.2.5}$$

where $i \leq 1$, $i_0 \leq 1$ and $i + i_0 + \sum_{k=1}^{n} j_k \leq m$.

Theorem A.2.3. (Itô's Formula) [50, 67] *Let $V \in C[R_+ \times R^n, R^m]$, and V_t, V_x, and V_{xx} exist and be continuous for $(t, x) \in R_+ \times R^n$, where V_x is an $m \times n$ Jacobian matrix of $V(t, x)$ and $V_{xx}(t, x)$ is an $n \times n$ Hessian matrix whose elements $\frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}$ are m-dimensional vectors. Let $y(t)$ be a solution process of* (4.1.1). *Then*

$$\begin{aligned} dV(t, y(t)) = V_t(t, y(t)) &+ V_x(t, y(t))\widehat{F}(t, y(t)) \\ &+ \frac{1}{2} tr(V_{xx}(t, y(t))\sigma(t, y(t))\sigma^T(t, y(t)))dt \\ &+ V_x(t, y(t))\sigma(t, y(t))dz(t). \end{aligned} \tag{A.2.6}$$

An important technique in the theory of random differential inequalities is concerned with estimating a random function satisfying differential inequality by extremal solutions of a corresponding random differential equation. One of the results that has wide range applicability is the following result.

Theorem A.2.4. [59] *Assume that $g \in M[R_+ \times R^m, R[\Omega, R^m]]$ and $g(t, u, \omega)$ is sample continuous and quasi-monotone non-decreasing in u w.p. 1 for fixed t;* (ii) *$r(t, \omega)$ is the maximal sample solution process of the random differential equation*

$$u' = g(t, u, \omega), \qquad u(t_0, \omega) = u_0(\omega) \tag{A.2.7}$$

existing for $t \geq t_0$;

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

(iii) $w \in C[R_+, R[\Omega, R^m]]$,

$$D^+ w(t,\omega) \leq g(t, w(t,\omega), \omega), \qquad \text{a.e. in } t, \tag{A.2.8}$$

and

$$w(t_0, \omega) \leq u_0 \qquad \text{w.p. } 1.$$

Then

$$w(t,\omega) \leq r(t,\omega) \qquad \text{for } t \geq t_0. \tag{A.2.9}$$

For any $n \times n$ random matrix process

$$\lim_{h \to 0} \frac{1}{h}[\|I + hA(t,\omega)\| - 1] \qquad \text{w.p. } 1 \tag{A.2.10}$$

is called the logarithmic norm of $A(t,\omega)$ and is denoted by $\mu(A(t,\omega))$. The following result gives properties of logarithmic norm.

Lemma A.2.3. **[60]** *Let $A(t,\omega)$ and $B(t,\omega)$ be $n \times n$ random matrix functions defined on a complete probability space $(\Omega, \mathcal{F}, P)$. The logarithmic norm possesses the following properties:*

(i) $\mu(\alpha A(t,\omega)) = \alpha \mu(A(t,\omega))$, *if* $\alpha \geq 0$;

(ii) $|\mu(A(t,\omega))| \leq \|A(t,\omega)\|$;

(iii) $\mathrm{Re}\,\lambda(t,\omega) \leq \mu(A(t,\omega))$, *where* $\lambda(t,\omega)$ *is an eigenvalue of* $A(t,\omega)$;

(iv) $\mu(A(t,\omega) + B(t,\omega)) \leq \mu(A(t,\omega)) + \mu(B(t,\omega))$;

(v) $|\mu(A(t,\omega)) - \mu(B(t,\omega))| \leq \|A(t,\omega) - B(t,\omega)\|$;

(vi) $\mu(A(t,\omega))$ *preserves the basic regularity properties of* $A(t,\omega)$.

Remark A.2.1. **[60]** Suppose that the hypotheses of Lemma A.2.2 hold. Then

$$\|x(t,t_0,y_0(\omega)) - x(t,t_0,z_0)\| \leq$$
$$\|y_0(\omega) - z_0\| \int_0^1 \left[\exp\left[\int_{t_0}^t \mu(\widehat{F}_y(u, x(u,t_0,z_0 + s(y_0(\omega) - z_0))))du\right]\right] ds. \tag{A.2.11}$$

In the following, we present certain type of class of monotone functions that will be utilized in the stability and error analysis.

Definition A.2.1. **[67]** A function ϕ is said to belong to the class $\mathcal{K}$ if $\phi \in C[[0,\rho), R_+]$, $\phi(0) = 0$, and $\phi(u)$ is strictly increasing in u, where $0 < \rho \leq \infty$.

Definition A.2.2. **[67]** A function b is said to belong to the class of $\mathcal{VK}$ if $b \in C[[0,\rho), R_+]$, $b(0) = 0$, and $b(u)$ is convex and strictly increasing in u.

Definition A.2.3. **[67]** A function $a(t,u)$ is said to belong to the class of $\mathcal{CK}$ if $a \in C[R_+ \times [0,\rho), R_+]$, $a(t,0) \equiv 0$, and $a(t,u)$ is concave and strictly increasing in u for each $t \in R_+$.

Definition A.2.4. **[67]** A function β is said to belong to the class of $\mathcal{L}$ if $\beta \in C[R_+, R_+]$, $\beta(t)$ is monotone decreasing in t, and $\beta(t) \to 0$ as $t \to \infty$.

Lemma A.2.4. **[83]** *Let $m, \lambda \in C[[t_0, t_0 + a), R_+]$, where R_+ denotes the non-negative real line. Suppose further that, for some non-negative constant C, we have*

$$m(t) \leq C + \int_{t_0}^{t} \lambda(s)m(s)ds, \qquad t \in [t_0, t_0 + a). \tag{A.2.12}$$

Then

$$m(t) \leq C \exp\left[\int_{t_0}^{t} \lambda(s)ds\right], \qquad t \in [t_0, t_0 + a). \tag{A.2.13}$$

Theorem A.2.5. **[83]** *Let $m, \nu \in C[[t_0, t_0 + a), R_+]$, $\eta \in C[[t_0, t_0 + a), R]$, and satisfy the inequality*

$$m(t) \leq \eta(t) + \int_{t_0}^{t} \nu(s)m(s)ds, \qquad t \in [t_0, t_0 + a). \tag{A.2.14}$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

Then

$$m(t) \leq \eta(t) + \int_{t_0}^{t} \nu(s)\eta(s) \left(\exp\left[\int_{s}^{t} \nu(u)du \right] \right) ds. \tag{A.2.15}$$

Remark A.2.2. We note that Lemma A.2.4 and Theorem A.2.5 remain valid, if m, ν and η are Lebesgue measurable and locally Lebesgue integrable on $[t_0, t_0 + a)$.

Lemma A.2.5. (Schwartz inequality) [31] *Let $x, y \in L^2[\Omega, R^n]$. Then*

$$(E[x^T(\omega)y(\omega)])^{1/2} \leq \|x\|_2 \|y\|_2$$

where $\|z\|_2 = \left(\int_\Omega \|z(\omega)\|^2 p(d\omega)\right)^{1/2}$.

The following results provide necessary and sufficient conditions for uniform stability and asymptotic stability of the trivial solution of (2.1.2).

Theorem A.2.6. [56] *The trivial solution of* (2.1.2) *is uniformly stable if and only if there exists a function $a \in \mathcal{K}$ verifying the estimate*

$$\|x(t, t_0, m_0)\| \leq \alpha(\|m_0\|), \qquad t \geq t_0 \tag{A.2.16}$$

for $\|m_0\| < \rho$.

Theorem A.2.7. [56] *The trivial solution of* (2.1.2) *is uniformly asymptotically stable if and only if there exist function $\alpha \in \mathcal{K}$, $\beta \in \mathcal{L}$ such that*

$$\|x(t, t_0, m_0)\| \leq \alpha(\|m_0\|)\sigma(t - t_0), \qquad t \geq t_0 \tag{A.2.17}$$

for $\|m_0\| < \rho$.

Theorem A.2.8. [82] *Let us consider a scalar differential equation,*

$$\nu' = \lambda(t,\omega)h(\nu) + \beta(t,\omega), \qquad \nu(t_0,\omega) = \nu_0(\omega) \tag{A.2.18}$$

where $\lambda, \beta \in C[R_+, R[\Omega, R_+]]$, $h \in C[R_+, R_+]$, $h \in \mathcal{K}$. *Then any sample solution process satisfies the following bound:*

$$\nu(t,\omega) \leq H^{-1}\left(\int_{t_0}^{t} \lambda(s,\omega)ds + H\left(\nu_0 + \int_{t_0}^{t} \beta(s,\omega)ds\right)\right) \tag{A.2.19}$$

for $t \geq t_0$ *w.p.* 1, *where* H *is defined by*

$$H(u) = \int_{u_0}^{u} \frac{ds}{h(s)}\, ds, \qquad \text{for } u_0, u > 0.$$

The following well-known Lebesgue dominated convergence theorem provides sufficient conditions under which the operations of limit and integration can be interchanged.

Theorem A.2.9. [**39, 98**] *Let* $(X, \mathcal{A}, \mu)$ *be a complete measure space. Let* g *be an integrable function over a non-empty set* $E \subseteq X$ *and* $\{f_n\}$ *be a sequence of measurable functions such that*

$$|f_n(x)| \leq g(x)$$

on E *and for almost all* x *in* E *we have*

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Then

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f\, d\mu.$$

The following result provides a method of finding a probability density function (pdf) of a function of random vectors in R^n. In fact, it is a change of variable technique in the theory of integration of functions of several variables.

Theorem A.2.10. [47] *Let X be an N-dimensional random vector with probability density function $f_X(x)$. Let W be a transformation/function defined on R^N into itself such that W is invertible and its inverse W^{-1} is continuously differentiable with respect to x. Then the probability density function of $Y = W(X)$ is given by*

$$f_Y(y) = f_X(W^{-1}(y))|J(y)|, \tag{A.2.20}$$

where

$$J(y) = \det\left(\frac{\partial W^{-1}(y)}{\partial y}\right)$$

and $|J(y)|$ is the absolute value of the determinant of the $N \times N$ Jacobian matrix of W^{-1}.

A.3. BOUNDARY VALUE PROBLEMS

In the following we present a fixed point theorem and a result concerning a measurable selection of multivalued maps.

Theorem A.3.1. (Schauder's Fixed Point Theorem) [28] *Let X be a real Banach space and C be a closed, bounded and convex subset of X. Let F be a compact map from C into itself. Then F has a fixed point.*

Theorem A.3.2. [28] *Let $(\Omega, \mathcal{F})$ be a measurable space. Let (X, d) be a separable metric space, $\mathcal{B}$ be the Borel σ-algebra of (X, d), and S be a multivalued map defined on Ω into $2^X\backslash\{\phi\}$ such that*

(i) *$S(\omega)$ is complete for all $\omega \in \Omega$;*

(ii) *$\rho(x, S(\cdot))$ is measurable for every $x \in X$, where $\rho(x, S)$ is the distance from x to S.*

Then S admits an $(\mathcal{F}, \mathcal{B})$-measurable selection, i.e., there exists a map g from Ω into X such that $g^{-1}(B) \in \mathcal{F}$ for $B \in \mathcal{B}$ and $g(\omega) \in S(\omega)$.

Under Caratheodory type of condition, we present a comparison theorem [67].

Theorem A.3.3. *Assume that*

(i) $g \in M[E, R[\Omega, R^n]]$ *and* $g(t, u, \omega)$ *is sample continuous in* u *for fixed* t, *where* $E = [t_0, t_0 + a) \times D$ *and* D *is an open set in* R^n;

(ii) $g(t, u, \omega)$ *is almost surely sample quasi-monotone non-decreasing in* u *for each* t;

(iii) $r(t, \omega)$ *is the sample maximal solution of the random differential system*

$$u' = g(t, u, \omega), \qquad u(t_0, \omega) = u_0(\omega) \tag{A.3.1}$$

existing on $[t_0, t_0 + a)$;

(vi) $z \in C[[t_0, t_0 + a), R[\Omega, \mathcal{R}^n]]$, $(t, z(t)) \in E$ *w.p.* 1,

$$z(t_0, \omega) \leq u_0(\omega) \qquad \text{w.p. } 1$$

and

$$D^+ z(t, \omega) \leq g(t, z(t, \omega), \omega) \qquad \text{a.e. } t \in [t_0, t_0 + a).$$

Then

$$z(t, \omega) \leq r(t, \omega) \qquad \text{for } t \in [t_0, t_0 + a). \tag{A.3.2}$$

Corollary A.3.1. *If in* Theorem A.3.3 *inequalities are reversed, then the conclusion of the* Theorem A.3.3 *is replaced by*

$$z(t, \omega) \geq \rho(t, \omega), \qquad t \in [t_0, t_0 + a) \tag{A.3.3}$$

where $\rho(t, \omega)$ *is the sample minimal solution process of* (A.3.1).

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

REFERENCES

1. Alekseev, V. M., An estimate for the perturbation of the solutions of ordinary differential equations (Russian) *Vestnik Moskov. Univ. Ser. I. Mat. Meh.*, no. 2 (1961) 28–36.

2. Arnold, L., *Stochastic Differential Equations: Theory and Applications*, Wiley, New York, 1974.

3. Bernfeld, S. and Chandra, Jagdish, Minimal and Maximal Solutions of Nonlinear Boundary Value Problems, *Pacific J. Math.* **71** (1977) 13–20.

4. Bernfeld, S. and Lakshmikantham, V., *An Introduction to Nonlinear Boundary Value Problems*, Academic Press, New York, 1974.

5. Bharucha-Reid, A. T., Random algebraic equations, in *Probabilistic Methods in Applied Mathematics* (A. T. Bharucha-Reid, Ed.) Vol. 2, 1–52, Academic Press, New York, 1970.

6. Bharucha-Reid, A. T., *Random Integral Equations*, Academic Press, New York, 1972.

7. Bharucha-Reid, A. T., Fixed point theorems in probabilistic analysis, *Bull. Amer. Math. Soc.* **82** (1976) 641–657.

8. Bharucha-Reid, A. T. and Sambandham, M., *Random Polynomials*, Academic Press, New York, 1986.

9. Boyce, W. E., Random Eigenvalue Problems, in *Probabilistic Methods in Applied Mathematics* (A. T. Bharucha-Reid, Ed.) Vol. 1, Academic Press, New York, 1968.

10. Boyce, W. E., Applications of the Liouville Equation, in *Approximate Solution of Random Equations* (A. T. Bharucha-Reid, ed.), North Holland, New York (1979) 13–25.

11. Bucy, R. S., Stability and Positive Supermartingales, *J. Diff. Equ.* **1** (1965) 151–155.

12. Cameron, A., *The Principles of Lubrication*, Wiley, New York, 1966.

13. Capocelli, R. M. and Ricciardi, L. M., A diffusion model for population growth in a random environment, *Theor. Pop. Biol.* **5** (1974) 28–41.

14. Castaing, C. and Valadies, M., Convex Analysis and Measurable Multifunctions, *Lect. Notes. Math.* **580**, Springer Verlag, 1977.

15. Chandra, Jagdish and Davis, Paul, A monotone method for quasilinear boundary value problem, *Arch. Rat. Mech. Anal.* **54** (1974) 257–266.

16. Chandra, Jagdish and Davis, Paul, Some analytic observations on the gas-lubricated slider bearing, *ASME J. Lubrication Technology* **104** (1982) 271–274.

17. Chandra, Jagdish, Ladde, G. S. and Lakshmikantham, V. On the fundamental theory of nonlinear second order stochastic boundary value problems, *Stoch. Anal. Appl.* **1** (1983) 1–19.

18. Changdra, J., Ladde, G. S. and Lakshmikantham, V., Stochastic analysis of compressible gas lubricated slider bearing problem, *SIAM J. on Appl. Math.* **43** (1983) 1174–1186.

19. Chandra, Jagdish and Ladde, G. S., On Roughness Effects in a Compressible Lubrication Problem, *Recent Developments in Applied Mathematics*, Rensselaer Press, New York, 8–12, 1983/84.

20. Chandra, Jagdish and Ladde, G. S., Comparison Theorems and Stochastic Boundary Value Problems, *Proceedings of the International Conference on Trends in Theory and Practice of Nonlinear Differential Equations*, Arlington, Texas, Marcel Dekker, Inc., New York, 111–119, 1984.

21. Chandra, Jagdish, Lakshmikantham, V. and Leela, S., Comparison principle and theory of nonlinear boundary value problem, *Nonlinear Phenomena in Mathematical Sciences* (edited by V. Lakshmikantham) Academic Press, 241–248, 1982.

22. Chernov, L. A., *Wave Propagation in a Random Medium*, McGraw-Hill, New York, 1960.

23. Chow, P. L. and Saibel, E. A., On the roughness effect in Hydrodynamic Lubrication, *Trans. of the ASME* **100** (1978) 176–180.

24. Chow, P. L. and Saibel, E. A., Analysis of a Stochastic Reynolds Equation and Related Problems, *Constructive Approach to Mathematical Models*, Academic Press, New York (1979) 321–326.

25. Christensen, M., Stochastic Models for Hydrodynamic Lubrication of Rough Surfaces, *Pract. Instn. Mech. Eng. Tribology Group* **184** (1969–70) 1013–1023.

26. Christensen, M. J. and Bharucha-Reid, A. T., Stability of the roots of random algebraic polynomials, *Commun. Stat. B: Simul. Comput.* **9** (1980) 179–192.

27. Chung, K. L., *A Course in Probability Theory*, 2nd ed. Academic Press, New York, 1974.

28. Deimling, K., *Nonlinear Functional Analysis*, Springer Verlag, 1985.

29. Deimling, K., Ladde, G. S. and Lakshmikantham, V., Sample Solutions of Stochastic Boundary Value Problem, *Stochastic Analysis and Applications* **3** (1985) 153–162.

30. Deo, S. G. and Lakshmikantham, V., *Method of Variation of Parameters for Dynamic Systems*, Gordon and Breach Science Publishers, Amsterdam, 1998.

31. Doob, J. L., *Stochastic Processes*, Wiley, New York, 1953.

32. Farahmand, K., *Topics in Random Polynomials*, Pitman Research Notes in Mathematics, Addison Wesley Longman, London, 1998.

33. Feldman, M. W. and Roughgarden, J., A population's stationary distribution and chance of extinction in a stochastic environment with remarks on the theory of species packing, *Theor. Pop. Biol.* **7** (1975) 197–207.

34. Fleming, W. H. and Rishel, R. W., *Deterministic and Stochastic Optimal Control*, Springer-Verlag, New York, 1975.

35. Freund, John E., *Mathematical Statistics*, Princeton-Hall, Englewood Cliffs, NJ, 1992.

36. Friedman, A., *Stochastic Differential Equations and Applications*, Vols. I and II, Academic Press, New York, 1975.

37. Fuller, Wayne A., *Introduction to Statistical Time Series*, John Wiley and Sons, New York, 1976.

38. Gause, G. F., *The Struggle for Existence*, Hafner, New York, 1934.

39. Gikhman, I. I. and Skorokhod, A. V., *Introduction to the Theory of Random Processes*, Saunders, Philadelphia, Pennsylvania, 1969.

40. Gikhman, I. I. and Skorokhod, A. V., *Stochastic Differential Equations*, Springer-Verlag, Berlin and New York, 1972.

41. Goel, N. S. and Richter-Dyn, N., *Stochastic Models in Biology*, Academic Press, New York, 1974.

42. Gray, K. B., The Applications of Stochastic Approximations to the Optimization of Random Circuits, *Proc. of Sympos. Appl. Math.* **6** (1964) 178–192, Amer. Math. Soc., Providence, RI.

43. Hartman, P., *Ordinary Differential Equations*, John Wiley & Sons, Inc., New York, 1964.

44. Hamblen, J. W., Distribution of roots of quadratic equations with random coefficients, *Ann. Math. Stat.* **27** (1956) 1136–1143.

45. Hoffman, W. C., Electromagnetic field in a randomly inhomogeneous medium, *IRE. Trans.* **AP-7**, Special Suppl. (1959) 5301–5306.

46. Hoffman, W. C., Wave propagation in a general random continuous medium, *Proc. of Sympos. Appl. Math.* **16** (1964) 117–144, Amer. Math. Soc., Providence, RI.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

47. Hogg, R. V. and Craig, P. T., *Introduction to Mathematical Statistics*, Fourth Edition, Macmillian Publishing Company, Inc., New York, 1978.

48. Keller, J. B., Stochastic Equations and Wave Propagation in Random Media, *Proc. Sympos. Appl. Math.* **16** (1964) 145–170, Amer. Math. Soc., Providence, RI.

49. Keller, H. B., Elliptic Boundary Value Problems Suggested by Nonlinear Diffusion Processes, *Arch. Rational Mech. Anal.* **35** (1969) 363–381.

50. Khas'minskii, R. Z., Ustoychivost'sistem differentsial'nykh uravneniy pri sluchaynykh vozmusheniyakh (Stability of systems of differential equations in the presence of random disturbances) Moscow, Nauka Press, 1969.

51. Kohler, W. E. and Boyce, W. E., A Numerical Analysis of Some First Order Stochastic Initial Value Problems, *SIAM J. Appl. Math.* **27** (1974) 167–179.

52. Kolmanovski, V. B. and Nosov, V. R., *Stability of Functional Differential Equations*, Academic Press, New York, 1986.

53. Kozin, F., On the probability densities of the output of some random systems, *J. Appl. Mech.* **28** (1961) 161–165.

54. Krengel, U. and Sucheston, L., Semiamarts and finite values, *Bull. Am. Math. Soc.* **83** (4) (1977) 745–747.

55. Kulkarni, R. M. and Ladde, G. S., Stochastic Perturbations of Nonlinear Systems of Differential Equations, *J. Math. and Phy. Sci.* **10** (1976) 33–45.

56. Ladde, G. S., Stability properties of functional differential equations, *J. Math. and Phys. Sci.* **4** (1970) 237–254.

57. Ladde, G. S., Variational comparison theorem and perturbations of nonlinear systems, *Proc. of Amer. Math. Soc.* **52** (1975) 181–187.

58. Ladde, G. S., Systems of differential inequalities and stochastic differential equations II, *J. Math. Phys.* **16** (1975) 894–900.

59. Ladde, G. S., Systems of differential inequalities and stochastic differential equations III, *J. Math. Phys.* **17** (1976) 2113–2120.

60. Ladde, G. S., Logarithmic norm and stability of linear systems with random parameters, *Int. J. Systems Sci.* **8** (1977) 1057–1066.

61. Ladde, G. S., Stability Technique and Thought Provocative Dynamical Systems, *Proceedings of the Conference on Nonlinear Systems and Applications*, Arlington, Texas, Academic Press, New York, 211–218, 1977.

62. Ladde, G. S., Stability Technique and Thought Provocative Dynamical Systems II, *Proceedings of the International Conference on Applied Nonlinear Analysis*, Arlington, Texas, Academic Press, New York, 215–218, 1979.

63. Ladde, G. S., Stochastic Boundary Value Problems with Applications, *J. of Math. and Phy. Sci.* **18** (1984) 53–63.

64. Ladde, G. S., Stochastic Delay Differential Systems, *Proceedings of the International Symposium on "Functional Differential Equations,"* Kyoto, Japan, World Scientific Publishing Co., New Jersey, 204–212, 1991.

65. Ladde, G. S., Modeling of Dynamic Systems by Itô-Type Systems of Stochastic Differential Equations, *Proceedings of the Second International Conference on "Integral Methods in Science and Engineering,"* Arlington, Texas, Hemisphere Publishing Corp., Washington, DC, 63–74, 1991.

66. Ladde, G. S., Hybrid systems: Convergence and stability analysis of large-scale approximations schemes, *Int. J. Hybrid Systems* **2** (2002) 235–260.

67. Ladde, G. S. and Lakshmikantham, V., *Random Differential Inequalities*, Academic Press, New York, 1980.

68. Ladde, G. S., Lakshmikantham, V. and Leela, S., A technique in perturbation theory, *Rocky Mount. J. Math.* **6** (1976) 133–140.

69. Ladde, G. S. and Lakshmikantham, V. and Sambandham, M., Comparison theorem and error estimates of stochastic differential systems, *Stoc. Anal. Appl.* **3** (1985) 23–62.

70. Laddc, G. S., Medhin, N. G. and Sambandham, M., Error estimates for random boundary value problems with applications to a hanging cable problem (in press).

71. Ladde, G. S. and Sambandham, M., Stochastic versus deterministic, *Math. Computers Simulation* **24** (1982) 507–514.

72. Ladde, G. S. and Sambandham, M., Error estimates of solutions and mean of solutions of stochastic differential systems, *J. Math. Physics* **24** (1983) 815–822.

73. Ladde, G. S., Sambandham, M. and Sathananthan, S., Comparison Theorem and Its Applications, *General Inequalities* **6**, Birkhäuser Verlag Basel, 321–342, 1992.

74. Ladde, G. S. and Sathananthan, S., Itô-type systems of stochastic integro-differential equations, *Proceedings of the Second International Conference on "Integral Methods in Science and Engineering,"* Arlington, Texas, Hemisphere Publishing Corp., Washington, DC, 75–89, 1991.

75. Ladde, G. S. and Sathananthan, S., Error Estimates and Itô-Type Systems of Nonlinear Stochastic Integro-Differential Equations, *Applicable Analysis* **43** (1992) 163–189.

76. Ladde, G. S. and Sathananthan, S., Stability of Lotka-Volterra Model, *Mathematical and Computer Modeling* **16** (1992) 99–107.

77. Ladde, G. S. and Sathananthan, S., Stochastic integro-differential equations with random parameters – I, *Dynamic Systems and Applications* **1** (1992) 369–390.

78. Ladde, G. S. and Sathananthan, S., Stochastic integro-differential equations with random parameters – II, *Dynamic Systems and Applications* **3** (1994) 563–582.

79. Ladde, G. S. and Robinson, J. V., A Stochastic Version of Turing's Cell Morphogenetic Model, *Biomathematics and Cell Kinetics* (Editor: M. Rotenberg) Elsevier/North-Holland: Biomedical Press, Amsterdam, The Netherlands, 349–356, 1981.

80. Ladde, G. S. and Robinson, V., Stability and limiting distributions of one and two species stochastic population models, *Mathematical Modelling* **5** (1984) 331–338.

81. Ladde, G. S. and Seikkala, S., On Sample Solutions of Random Initial Value and Nicoletti Boundary Value Problems, *Mathematics and Computers in Simulation* **29** (1987) 223–231.

82. Lakshmikantham, V., Generalization of Bellman-Gronwall-Reid Inequalities, *J. Math. Anal. Appl.* **41** (1973) 199–204.

83. Lakshmikantham, V. and Leela, S., *Differential and Integral Inequalities, Theory and Applications*, Vol. I, Academic Press, New York, 1969.

84. Lax, M. D., The Method of Moments for Linear Random Boundary Value Problems, *SIAM J. Appl. Math.* **31** (1976) 62–83.

85. Lax, M. D., Obtaining Approximate Solutions of Random Differential Equations by Means of the Method of Moments, *Approximate Solutions of Random Equations* (A. T. Bharucha-Reid, Ed.) North-Holland, New York, 127–147, 1979.

86. Lax, M. D. and Boyce, W. E., The Method of Moments for Linear Random Initial Value Problems, *J. Math. Anal. Appl.* **53** (1976) 111–132.

87. Levins, R., The effect of random variations of different types on population growth, *Proc. Nat. Acad. Sci.* **62** (1969) 1061–1065.

88. Levontin, R. C. and Cohen, D., On population growth in a randomly varying environment, *Proc. Nat. Acad. Sci.* **62** (1969) 1056–1060.

89. Lord, M. E. and Mitchell, A. R., A new approach to the method of nonlinear variation of parameters, *Appl. Math. Computation* **4** (1978) 95–105.

90. Lotka, A. J., *Elements of Mathematical Biology*, Dover Publishing, New York, 1956.

91. May, R. M. *Stability and Complexity in Model Ecosystems*, Princeton University Press, Princeton, 1973.

92. Masse, P., *Optimal Investment Decisions*, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

93. Mönch, H., Boundary Value Problems for Nonlinear Ordinary Differential Equations of Second Order in Banach Spaces, *Nonlinear Analysis* 4 (1980) 985–999.

94. Montroll, E. W., Some statistical aspects of the theory of interacting species, *Some Mathematical Questions in Biology* III, Amer. Math. Soc. (1972) 101–143.

95. Morozan, T., *Stabilities sistemelor cu Parametri Aleatori*, Editura Academiei Republicii Socialiste România, Bucharest, 1969.

96. Nashed, M. Z. and Engl, H. W., *Random Generalized Inverses and Approximate Solutions of Random Operator Equations* (A. T. Bharucha-Reid, Ed.), North Holland, New York (1979) 149–200.

97. Odum, E. D., *Fundamentals of Ecology*, 2nd Edition, Saunders, Philadelphia, 1959.

98. Royden, N. L., *Real Analysis*, 3rd Edition, Prentice Hall, Englewood Cliffs, 1988.

99. Padgett, W. J., Schultz, G. and Tsokos, C. P., A random differential equation approach to the probability distribution of BOD and DO in streams, *SIAM J. Appl. Math.*, **32** (1977) 467–483.

100. Powers, D. L., *Boundary Value Problems*, 2nd Edition, Academic Press, New York, 1979.

101. Siljak, D. D., *Large Scale Dynamic Systems: Stability and Structure*, North-Holland, 1978.

102. Soong, T. T. and Chuang, S. N., Solutions of a Class of Random Differential Equations, *SIAM J. Appl. Math.* **24** (1973) 449–459.

103. Soong, T. T., *Random Differential Equations in Science and Engineering*, Academic Press, New York, 1973.

104. Soong, T. T., Pharmacokinetics with Uncertainties in Rate Constants, *Math. Biosci.* **12** (1971) 235–243.

105. Šparo, D. I. and Šur, M. G., On the distribution of roots of random polynomials, Vestn. Mosk. Univ., Ser. 1: Mat. Mekh. (1962) 40–53.

106. Sun, D. C. and Chen, K. K., First effects of Stokes roughness on hydrodynamic lubrication, *ASME J. Lubrication Technology* **99** (1977) 2–9.

107. Sussmann, H. J., On the Gap Between Deterministic and Stochastic Ordinary Differential Equations, *The Annals of Probability* **6** (1978) 19–41.

108. Tsokos, C. P. and Padgett, W. J., *Random Integral Equations with Applications to Life Science and Engineering*, Academic Press, New York, 1974.

109. vom Scheidt, J. and Bharucha-Reid, A. T., On the averaging problem for the roots of random algebraic polynomials, *Wiss. Beitr. Ingenieunhochsch. Zwickau, Sonderh.* (1983) 1–43.

110. Weding, W., Stochastic boundary and eigenvalue problems, *Stochastic Problems in Dynamics* (B. L. Clarkson, Ed.) Fearon, Pitman Pub., 54–66, 1977.

111. White, J. W., Surface roughness on the load carrying capacity of very thin compressible lubricating films, *ASME J. of Lubrication Tech.* **102** (1980) 445–451.

112. White, B. S. and Franklin, J. N., A Limit Theorem for Stochastic Two-Point Boundary Value Problems of Ordinary Differential Equations, *Comm. Pure Appl. Math* **32** (1979) 253–275.

113. Xia, N. M., Boyce, W. E. and Barry, M. R., Two Point Boundary Value Problems Containing a Finite Number of Random Variables, *Stochastic Annal. Appl.* **1** (1983) 117–137.

114. Xia, Ning-Mao and Boyce, W. E., The Approach to Normality of the Solutions of Random Boundary and Eigenvalue Problems with Weakly Correlated Coefficients, *Quart. Appl. Math.* **40** (1983) 419–445.

115. Xia, Ning-Mao and Boyce, W. E., The Density Function of the Solution of a Random Initial Value Problem Containing Small Stochastic Processes, *SIAM J. Appl. Math.* **44** (1984) 1192–1209.

116. Xia, Ning-Mao, The Solutions for the Two-Point Boundary Value Problems of Stochastic Differential Equations Containing Small White Noise, *Acta Math. Appl.* **8** (1985) 340–350.

117. Xia, Ning-Mao, The Density Function of the Solution of a Two-Point Boundary Value Problem Containing Small Stochastic Processes, *Quart. Appl. Math.* **46** (1988) 29–47.

118. Girko, V. L., *Theory of Random Determinants*, Kluwer Academic Publishers, Boston, 1990.

119. Ince, E. L., *Ordinary Differential Equations*, New York: Dover, 1926 (1st Edition), 1956.

120. Pielou, E. C., *An Introduction to Mathematical Ecology*, Wiley Interscience, NY, 1969.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

INDEX

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016